Mathematics

A basic course for beginners in G. C. E. (Advanced Level) Mathematics

Department of Mathematics
Faculty of science and Technology
National Institute of Education
Maharagama
Sri Lanka

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Director General's Message

This book " A basic course for beginners in G.C.E. (Advanced Level) Mathematics " is written for the pupils who prepare to continue their studies in G.C.E. (A.L) Mathematics stream after their G.C.E. (Ordinary Level) examination.

The salient feature of this book is giving a clear understanding, basic knowledge in G.C.E. (A.L) Mathematics and self-confidence to follow the subject.

Every chapter in this book is written by the writers with the precaution of national curriculum to offer clarity and richues. This book will help the students as a self-learning guide and to grasp the subject quickly and easily.

I am inclined to believe that this book will be found equally useful to both pupils and teachers.

I hope that the mathematics department will publish such books in future as well.

Upali M Seedera
Director General
National Institute of Education

PREFACE

This book " A basic course for beginners in G.C.E. (Advanced Level) Mathematics " is written specifically to meet the requirements for the pupils, who like to continue their studies in Mathematics or Combined Mathematics for G.C.E. (Advanced Level). Whole fundamental principles are emphasised, and attention is paid on basic mathematical problems and concepts, to make the pupils understand and practise in exercise.

One of the most important feature of this book is that it has been written for self study to the pupils expecting the results of the G.C.E. (Ordinary Level) examination.

The striking feature of the book is a number of solved problems, which are given to motivate the pupils for selflearning. More attention is focused on Algebra in this book.

However, suggestions or comments for the improvement of this book including criticisms if any, will be welcomed and incorporated in the subsequent edition.

Director
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Mathematics
Content

Algebra

Geometry

1. Algebra

1. Binomial Expansion

We learnt the following expansions in G.C.E. (O/L)

$(a+b)^2$, $(a-b)^2$, $(a+b)^3$ and $(a-b)^3$

Now we will obtain the above binomial expansions.

$(a+b)^2 = (a+b)(a+b)$

$= a(a+b) + b(a+b)$

$= (a^2 + ab + ab + b^2$

$= a^2 + 2ab + b^2$ ——————— (1)

$(a-b)^2 = (a-b)(a-b)$

$= a(a-b) - b(a-b)$

$= a^2 - ab - ab + b^2$

$= a^2 - 2ab + b^2$ ——————— (2)

OR

The result (2) above can be obtained by using the result (1).

$(a-b)^2 = [a+(-b)]^2$

Replacing b, by $(-b)$ in the result (1)

ie $(a+b)^2 = a^2 + 2ab + b^2$

$[a + (-b)]^2 = a^2 + 2a(-b) + (-b)^2$

$(a-b)^2 = a^2 - 2ab + b^2$

$(a+b)^3 = (a+b)(a+b)^2$

$= (a+b)(a^2 + 2ab + b^2)$

$= a(a^2 + 2ab + b^2) + b(a^2 + 2ab + b^2)$

$= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3$

$= a^3 + 3a^2b + 3ab^2 + b^3$ ——————— (3)

$(a-b)^3 = (a-b)(a-b)^2$

$= (a-b)[a^2 - 2ab + b^2]$

$= a(a^2 - 2ab + b^2) - b(a^2 - 2ab + b^2)$

$= a^3 - 2a^2b + ab^2 - a^2b + 2ab^2 - b^3$

$= a^3 - 3a^2b + 3ab^2 - b^3$ ——————— (4)

OR

The expansion of $(a - b)^3$ can be obtained by replacing b by $(-b)$ in (3).

$$(a + b)^3 = a^3 + 3a^2 b + 3a b^2 + b^3$$

$$[a + (-b)]^3 = a^3 + 3a^2(-b) + 3a(-b)^2 + (-b)^3$$

$$= a^3 - 3a^2 b + 3a b^2 - b^3$$

$$(a + b + c)^2 = [(a + b) + c]^2$$

$$= (a + b)^2 + 2(a + b)c + c^2$$

$$= a^2 + 2ab + b^2 + 2ac + 2bc + c^2$$

$$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

$(a + b)^2 = a^2 + 2ab + b^2$
$(a - b)^2 = a^2 - 2ab + b^2$
$(a + b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$
$(a - b)^3 = a^3 - 3a^2 b + 3ab^2 - b^3$
$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

We will apply the above results in solving related problems.

Example 1

Expand the following binomials.

(i) $(2x + 3y)^2$ (ii) $(2xy - 5z)^2$

(iii) $(3x + 2y)^3$ (iv) $\left(ab - \frac{2}{c}\right)^3$

(v) $(a + b - c)^2$

(i) $(2x + 3y)^2 = (2x)^2 + 2 \times 2x \times 3y + (3y)^2$

$$= 4x^2 + 12xy + 9y^2$$

(ii) $(2xy - 5z)^2 = (2xy)^2 - 2 \times 2xy \times 5z + 3(5z)^2$

$$= 4x^2y^2 - 20xyz + 25z^2$$

(iii) $(3x + 2y)^3 = (3x)^3 + 3 \times (3x)^2 \times (2y) + 3 \times (3x) \times (2x)^2 + (2y)^3$

$$= 27x^3 + 54x^2y + 36xy^2 + 8y^3$$

(iv) $\left(ab - \frac{2}{c}\right)^3 = (ab)^3 - 3(ab)^2 \times \frac{2}{c} + 3(ab)\left(\frac{2}{c}\right)^2 - \left(\frac{2}{c}\right)^3$

$$= a^3 b^3 - \frac{6^2 a^2 b^2}{c} + \frac{12ab}{c^2} - \frac{8}{c^3}$$

(v) $(a + b - c)^2 = a^2 + b^2 + c^2 + 2ab - 2bc - 2ca$

Example 2

Given that $a+b=4$ and $ab=5$ find the value of (i) a^2+b^2 and (ii) a^3+b^3

$$(a+b)^2 = a^2+2ab+b^2$$

$$a^2+b^2 = (a+b)^2 - 2ab$$

$$= 4^2 - 2\times 5 = 16-10=6$$

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

$$a^3+b^3 = (a+b)^3 - 3a^2b - 3ab^2$$

$$= (a+b)^3 - 3ab(a+b)$$

$$= 4^3 - 3\times 5\times 4$$

$$= 64-60 = 4$$

Exercise 1

Write the expansion of the following:

1. $(2a+3b)^2$
2. $(3a-4b)^2$
3. $\left(x+\frac{1}{x}\right)^2$
4. $(2xy+5z)^2$
5. $\left(\frac{1}{a}+\frac{1}{b}\right)^2$
6. $\left(x-\frac{1}{x}\right)^2$
7. $\left(\frac{a}{2}-\frac{2}{a}\right)^2$
8. $\left(\frac{1}{a}-\frac{2}{b}\right)^2$
9. $(4xy-3z)^2$
10. $(a+2b)^3$
11. $(2a-b)^3$
12. $(3a+2b)^3$
13. $\left(x+\frac{1}{x}\right)^3$
14. $\left(x-\frac{1}{x}\right)^3$
15. $(ab-2c)^3$
16. $\left(\frac{1}{a}+\frac{1}{b}\right)^3$
17. $\left(\frac{1}{a}-\frac{2}{b}\right)^3$
18. $(2xy-3z)^3$
19. $(a+b+c)^2$
20. $(a+b-c)^2$
21. $(a-b+c)^2$
22. $(a-b-c)^2$
23. $(a-2b+c)^2$
24. $(a-b-2c)^2$
25. Evaluate

   (i) 101^3 (ii) 198^3 (iii) 401^3 (iv) 999^3

26. Evaluate

   (i) $101^2 + 2\times 101\times 99 + 99^2$ (ii) $88^2 - 2\times 88\times 87 + 87^2$

27. **Evaluate**

(a) $51^3 + 3 \times 51^2 \times 49 + 3 \times 51 \times 49^2 + 49^3$

(b) $101^3 - 3 \times 101^2 \times 99 + 3 \times 101 \times 99^2 - 99^3$

28. Show that

(i) $(a+b)^2 + (a-b)^2 = 2(a^2 + b^2)$

(ii) $(a+b)^2 - (a-b)^2 = 4ab$

(iii) $(a+b)^3 + (a-b)^3 = 2a(a^2 + 3b^2)$

(i) $(a+b)^3 - (a-b)^3 = 2b(3a^2 + b^2)$

29. If $x + \frac{1}{x} = a$, find the values of (a) $x^2 + \frac{1}{x^2}$ and (b) $x^3 + \frac{1}{x^3}$ in terms of a.

30. If $x - y = 4$ and $xy = 21$, find the value of $x^3 - y^3$

31. If $(x + y) = -\frac{1}{3}$, find the value of $x^3 + y^3 - xy$

32. If $a - \frac{1}{a} = -5$, find the value of $a^3 - \frac{1}{a^3} - 200$

33. If $\frac{x^2 - 1}{x} = 4$, show that $\frac{x^6 - 1}{x^3}$ is 76.

34. If $\frac{a^2 - 1}{a} = 2$, find the value of $\frac{a^6 - 1}{a^3}$

35. If $a + b - 3 = 0$, find the value of $a^3 + b^3 + 9ab - 26$

36. If $a + b - 7 = 0$ and $ab = 12$, find the value of $a^3 + b^3 + 4ab(a + b)$

37. If $p = 2q + 4$, show that $p^3 - 8q^3 - 24pq = 64$

38. If $a + b + c = 0$, show that $a^3 + b^3 + c^3 = 3abc$

39. If $p + q = 1 + pq$, show that $p^3 + q^3 = 1 + p^3q^3$

40. If $ab(a + b) = p$, show that $a^3 + b^3 + 3p = \frac{p^3}{a^3b^3}$

2. Factorization

Factorization of algebraic expression

2.1 Trinomials

Examples for trinomials.

$x^2 - 5x - 6$, $2x^3 - 5x^2 - 3x$

$3x^2 - 5xy - 2y^2$

Example 1

Factorise : $x^2 - 5x - 6$

$= x^2 - 6x + x - 6$

$= x(x - 6) + 1(x - 6)$

$= (x - 6)(x + 1)$

Example 2

Factorise : $2x^3 - 5x^2 - 3x$

$= x\,[2x^2 - 5x - 3]$

$= x\,[2x^2 - 6x + x - 3]$

$= x\,[2x(x - 3) + 1(x - 3)]$

$= x\,[(x - 3)(2x + 1)]$

$= x(x - 3)(2x + 1)$

Example 3

Factorise : $3x^2 - 4xy - 4y^2$

$= 3x^2 - 6xy + 2xy - 4y^2$

$= 3x(x - 2y) + 2y\,(x - 2y)$

$= (x - 2y)(3x + 2y)$

Example 4

Factorise : $2(x + 3)^2 - 7(x + 3) - 4$

Let $x + 3 = a$

$= 2a^2 - 7a - 4$

$= 2a^2 - 8a + a - 4$

$= 2a(a - 4) + 1(a - 4)$

$= (2a + 1)(a - 4)$

$= [2(x + 3) + 1]\;[x + 3 - 4]$

$= (2x + 7)(x - 1)$

Example 5

Factorise : $2(2a+b)^2 - 5(2a+b)(a-2b) - 3(a-2b)^2$

Let $x = 2a + b$ and $y = a - 2b$

$= 2x^2 - 5xy - 3y^2$

$= 2x^2 - 6xy + xy - 3y^2$

$= 2x(x-3y) + y(x-3y)$

$= (x-3y)(2x+y)$

$= [(2a+b) - 3(a-2b)]\ [2(2a+b) + (a-2b)]$

$= (7b-a)5a = 5a(7b-a)$

2.2 Difference of squares.

$a^2 - b^2$

$a^2 - b^2 = a^2 - ab + ab - b^2$

$= a(a-b) + b(a-b)$

$= (a-b)(a+b)$

$$\boxed{a^2 - b^2 = (a-b)(a+b)}$$

Example 1

Factorise : $a^3b - ab^3$

$= ab(a^2 - b^2)$

$= ab(a-b)(a+b)$

Example 2

Factorise : $x^4 - 1$

$= (x^2)^2 - 1^2$

$= (x^2 - 1)(x^2 + 1)$

$= (x-1)(x+1)(x^2+1)$

Example 3

Factorise : $a^4 + 4b^4$

$a^4 + 4b^4 = a^4 + 4a^2b^2 + 4b^4 - 4a^2b^2$

$= (a^2 + 2b^2)^2 - (2ab)^2$

$= (a^2 + 2b^2 - 2ab)(a^2 + 2ab + b^2)$

Example 4

Factorise : $1 - a^2 + 2ab - b^2$

$= 1 - (a^2 - 2aab + b^2)$

$= 1^2 - (a-b)^2$

$= [1 - (a-b)]\ [1 + (a-b)]$

$= (1 - a + b)(1 + a - b)$

3. Factorising $a^3 + b^3$ and $a^3 - b^3$

Consider the product $(a + b)(a^2 - ab + b^2)$

$(a + b)(a^2 - ab + b^2)$

$= a(a^2 - ab + b^2) + b(a^2 - ab + b^2)$

$= a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3$

$= a^3 + b^3$

Therefore $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Consider the product $(a - b)(a^2 + ab + b^2)$

$(a - b)(a^2 + ab + b^2) = a(a^2 + ab + b^2) - b(a^2 + ab + b^2)$

$= a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3$

$= a^3 - b^3$

Therefore $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$$\boldsymbol{a^3 + b^3 = (a + b)(a^2 - ab + b^2)}$$
$$\boldsymbol{a^3 - b^3 = (a - b)(a^2 + ab + b^2)}$$

Example 1

Factorise: $81x^3 - 3y^3$

$= 3[27x^3 - y^3)$

$= 3[(3x)^3 - y^3]$

$= 3(3x - y)[(3x)^2 + 3x \times y + y^2]$

$= 3(3x - y)(9x^2 + 3xy + y^2)$

Example 2

Factorise: $x^3 + \frac{1}{x^3}$

$$x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)\left(x^2 - x \times \frac{1}{x} + \left(\frac{1}{x}\right)^2\right)$$

$$= \left(x + \frac{1}{x}\right)\left(x^2 - 1 + \frac{1}{x^2}\right)$$

Example 3

Factorise: $x^3 - \frac{1}{x^3}$

$$x^3 - \frac{1}{x^3} = \left(x - \frac{1}{x}\right)\left[x^2 + x \times \frac{1}{x} + \left(\frac{1}{x}\right)^2\right]$$

$$= \left(x - \frac{1}{x}\right)\left(x^2 + 1 + \frac{1}{x^2}\right)$$

Example 4

Factorise: $8a^3 + (b + c)^3$

$$8a^3 + (b + c)^3 = (2a)^3 + (b + c)^3$$
$$= [2a + (b + c)][(2a)^2 - 2a(b + c) + (b + c)^2]$$
$$= (2a + b + c)(4a^2 - 2ab - 2ac + b^2 + 2bc + c^2)$$
$$= (2a + b + c)(4a^2 + b^2 + c^2 - 2ab - 2ac + 2bc)$$

Example 5

Factorise: $a^3 - 27(b - c)^3$

$$a^3 - 27(b - c)^3 = a^3 - \{3(b - c)\}^3$$
$$= [a - 3(b - c)][a^2 + 3a(b - c) + 9(b - c)^2]$$
$$= (a - 3b + 3c)(a^2 + 9b^2 + 9c^2 + 3ab - 3ac - 18bc)$$

Example 6

(i) **Factorise:** $(a + b)^3 + c^3$

(ii) Write the expansion of $(a + b)^3$ and show that $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

(iii) **Using the above results factorise** $a^3 + b^3 + c^3 - 3abc$

(i) $(a + b)^3 + c^3 = [(a + b) + c][(a + b)^2 - c(a + b) + c^2]$
$$= (a + b + c)(a^2 + 2ab + b^2 - ac - bc + c^2)$$
$$= (a + b + c)(a^2 + b^2 + c^2 + 2ab - ac - bc)$$

(ii) $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a + b)^3 - 3a^2b - 3ab^2 = a^3 + b^3$

$(a + b)^3 - 3ab(a + b) = a^3 + b^3$

$a^3 + b^3 = (a + b)^3 - 3ab(a + b)$

(iii) $a^3 + b^3 + c^3 - 3abc$

$= (a+b)^3 - 3ab(a+b) + c^3 - 3abc$; [from (ii)]

$= (a+b)^3 + c^3 - 3ab(a+b) - 3bc$

$= [(a+b)+c][(a+b)^2 - c(a+b) + c^2] - 3ab[a+b+c]$; [from (i)]

$= (a + b + c)(a^2 + b^2 + c^2 + 2ab - ac - bc) - 3ab(a + b + c)$

$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

Exercise 2.1

Factorise:

1.	$x^2 - x - 6$	2.	$x^2 + 4x - 96$
3.	$x^2 + 5x - 6$	4.	$x^2 - 4x - 12$
5.	$x^2 + x - 42$	6.	$x^2 - 9x + 18$
7.	$2x^2 + 5x + 3$	8.	$2x^2 - 5x + 3$
9.	$2x^2 + 5x - 3$	10.	$2x^2 - 5x - 3$
11.	$10 - 7x - 12x^2$	12.	$15 + x - 2x^2$
13.	$18x^2 - 33x - 216$	14.	$6x^2 - 55x + 126$
15.	$2x^2 - 5xy + 3y^2$	16.	$6x^2 - 5xy - 6y^2$
17.	$4x^2 + 8xy + 3y^2$	18.	$2a^2 - 27ab + 13b^2$
19.	$40x^2y^2 + 49xy - 24$	20.	$32x^2 - 36xy - 35y^2$
21.	$24a^3 - 17a^2b - 20ab^2$	22.	$18a^3 - 3a^2b - 10ab^2$
23.	$(a^2 - 3a)^2 - 38(a^2 - 3a) - 80$	24.	$(a + b + c)^2 - 3(a + b + c) - 28$
25.	$2(x+y)^2 - 3(x+y) - 27$	26.	$2(2x+y) - 5(2x+y)(x-2y) + 3(x-2y)^2$
27.	$x^2 + x - (a-1)(a-2)$	28.	$x^2 - x - (a-1)(a-2)$
29.	$x^2 - \left(a + \frac{1}{a}\right)x + 1$	30.	$x^2 + 2ax + (a+b)(a-b)$
31.	$ax^2 + (ab-1)x - b$	32.	$x^2 + ax - (6a^2 - 5ab + b^2)$
33.	$4(a^2 - b^2)^2 - 8ab(a^2 - b^2) - 5a^2b^2$		
34.	$10(a+2b)^2 + 21(a+2b)(2a-b) - 10(2a-b)^2$		
35.	$6(x + y)^2 - 5(x^2 - y^2) - 6(x-y)^2$		

Exercise 2.2

Factorise:

1.	$x^2 - 4y^2$	2.	$x^3 - x$	3.	$x^2 - \frac{1}{x^2}$
4.	$x^5 - x$	5.	$4 - 9a^2$	6.	$(a-4b)^2 - 9b^2$
7.	$16 - (a+b)^2$	8.	$9 - (a-b)^2$	9.	$12a^3 - 3ab^2$
10.	$1 - (a-b)^2$	11.	$1 - (a+b)^2$	12.	$x^2 - y^2 - x - y$
13.	$x^2 - y^2 - x + y$	14.	$x^2 - y^2 + x + y$	15.	$x^2 - y^2 + x - y$
16.	$a^2 - b^2 - 4a + 4b$	17.	$a^2 - b^2 - 4a + 4$	18.	$ab + ac - (b+c)^2$
19.	$a(a+1) - b(b+1)$	20.	$x^4 - 3x^2y^2 + y^4$	21.	$x^4 + x^2y^2 + y^4$
22.	$a^4 + 5a^2b^2 + 9b^4$	23.	$x^2 - 4xy + 4y^2 - z^2$	24.	$4a^2 + b^2 - x^2 + 4ab$
25.	$x^4 + x^2 + 1$	26.	$4a^4 + 11a^2b^2 + 9b^4$		

Evaluate:

1.	$100^2 - 99^2$	2.	$94^2 - 36$	3.	$12.38^2 - 7.62^2$
4.	$6.2^2 - 3.8^2$	5.	$100 \times 99 + 1$	6.	$11.7 \times 9.3 + 8.3 \times 9.3$
7.	$\sqrt{148 \times 140 + 16}$	8.	$319^2 - 318 \times 320$	9.	$12.5^2 - 13 \times 12$
10.	103×97				

Exercise 2.3

Factorise:

1.	$a^3 + 8b^3$	2.	$27a^3 - b^3$	3.	$125a^3 - 64b^3$
4.	$8a^3b^3 - c^3$	5.	$x^3 + \frac{1}{x^3}$	6.	$x^3 - \frac{1}{x^3}$
7.	$\frac{1}{a^3} + \frac{1}{b^3}$	8.	$\frac{1}{a^3} - \frac{1}{b^3}$	9.	$a^3 + (b+c)^3$
10.	$a^3 + (b-c)^3$	11.	$a^3 - (b-c)^3$	12.	$8x^3 + (2y-x)^3$
13.	$(a+b)^3 + (a-b)^3$	14.	$(a+b)^3 - (a-b)^3$	15.	$8(a+b)^4 + (a+b)$
16.	$x^6 - y^6$	17.	$x^6 + y^6$	18.	$x^6 - 27$

19. (a) Factorise $(a+b)^3 + c^3$

(b) Show that $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

(c) Using the results in (a) and (b), factorise $a^3 + b^3 + c^3 - 3abc$

Hence, factorise the following polynomials.

(i) $x^3 + y^3 - z^3 + 3xyz$

(ii) $8x^3 + y^3 + z^3 - 6xyz$

(iii) $x^3 + 8y^3 - 27z^3 + 18xyz$

(iv) $a^3 - 28b^3 - 9ab^2$

(v) $8a^3 + b^3 - 1 + 6ab$

20. Show that

(i) If $a = b + c$, then $a^3 - b^3 - c^3 = 3abc$

(ii) If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$

(iii) If $z = 2x - 3y$, then $8x^3 - 27y^3 - x^3 = 18xyz$

21. Given that $x + y + z = 0$, show that $x^3 + y^3 + z^3 = 3xyz$

Hence, factorise the following algebraic polynomials.

(i) $(a-b)^3 + (b-c)^3 + (c-a)^3$

(ii) $(2x - 3y)^3 + (3y - 4z)^3 + 8(2z - x)^3$

(iii) $a^3(b-c)^3 + b^3(c-a)^3 + c^3(a-b)^3$

(iv) $(x - 3y)^3 + (3y - 4z)^3 + (4z - x)^3$

3. Algebraic Fractions

Lowest Common Multiple (L.C.M.)

Lowest common Multiple of simple polynomials can be easily found resolved into their elementary factors.

Example 1

Find the L.C.M. of $8x^3$, $12x^5$, and $18x^7$

$8x^3 = 2^3 \times x^3$

$12x^5 = 2^2 \times 3 \times x^5$

$18x^7 = 2 \times 3^2 \times x^7$

Hence L.C.M. is $2^3 \times 3^2 \times x^7 = 72x^7$

Example 2

Find the L.C.M. of $2x^2 - 8$, $3x^2 + 3x - 6$ and $6x^2 - 6x - 12$

$2x^2 - 8 = 2(x^2 - 4) = 2(x-2)(x+2)$

$3x^2 + 3x - 6 = 3(x^2 + x - 2) = 3(x+2)(x-1)$

$6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$

L.C.M. is $6(x-2)(x+2)(x-1)(x+1)$

Simplying Algebraic Fractions

Example 1 $\dfrac{2}{x^2-1} - \dfrac{3}{(x-1)^2}$

We have to find L.C.M. of $(x^2 - 1)$ and $(x-1)^2$

$x^2 - 1 = (x-1)(x+1)$

$(x-1)^2 = (x-1)^2$

L.C.M. is $(x-1)^2(x+1)$

$$\frac{2}{x^2-1} - \frac{3}{(x-1)^2}$$

$$= \frac{2}{(x-1)(x+1)} - \frac{3}{(x-1)^2}$$

$$= \frac{2(x-1) - 3(x+1)}{(x-1)^2(x+1)}$$

$$= \frac{-x-5}{(x-1)^2(x+1)}$$

$$= \frac{-(x+5)}{(x-1)^2(x+1)}$$

Example 2

Simplify.

$$\frac{2}{1+x}+\frac{1}{x-1}+\frac{3x}{1-x^2}$$

$$= \frac{2}{1+x}-\frac{1}{1-x}+\frac{3x}{1-x^2}$$

$$= \frac{2}{1+x}-\frac{1}{1-x}+\frac{3x}{(1-x)(1+x)}$$

$$= \frac{2(1-x)-(1+x)+3x}{(1-x)(1+x)}$$

$$= \frac{1}{(1-x)(1+x)}$$

Example 3

$$\frac{1}{x^2-4}+\frac{1}{x^2+x-6}-\frac{2}{x^2+5x+6}$$

$$= \frac{1}{(x-2)(x+2)}+\frac{1}{(x+3)(x-2)}-\frac{2}{(x+2)(x+3)}$$

$$= \frac{x+3+x+2-2(x-2)}{(x-2)(x+2)(x+3)}$$

$$= \frac{9}{(x-2)(x+2)(x+3)}$$

Example 4

$$\frac{3x}{2-3x+x^2}+\frac{4}{1-x}-\frac{6}{2-x}$$

$$= \frac{3x}{(2-x)(1-x)}+\frac{4}{1-x}-\frac{6}{2-x}$$

$$= \frac{3x+4(2-x)-6(1-x)}{(2-x)(1-x)}$$

$$= \frac{2+5x}{(2-x)(1-x)}$$

Example 5

$$\frac{a+2}{a-2}+\frac{4}{4-a^2}-1$$

$$=\frac{a+2}{a-2}-\frac{4}{a^2-4}-1$$

$$=\frac{a+2}{a-2}-\frac{4}{(a-2)(a+2)}-1$$

$$=\frac{(a+2)^2-4-(a-2)(a+2)}{(a-2)(a+2)}$$

$$=\frac{a^2+4a+4-4-(a^2-4)}{(a-2)(a+2)}$$

$$=\frac{4(a+1)}{(a-2)(a+2)}$$

Example 6

$$\frac{1}{4-\frac{3}{2+\frac{x}{1-x}}}$$

$$=\frac{1}{4-\frac{3}{\frac{2(1-x)+x}{1-x}}}$$

$$=\frac{1}{4-\frac{3}{\frac{2-x}{1-x}}}$$

$$=\frac{1}{4-\frac{3(1-x)}{2-x}}$$

$$=\frac{1}{\frac{4(2-x)-3(1-x)}{(2-x)}}=\frac{1}{\frac{8-4x-3+3x}{2-x}}$$

$$=\frac{1}{\frac{5-x}{2-x}} \qquad =\frac{2-x}{5-x}$$

Example 7

Simplify

$$\frac{x^2-25}{x^2+3x-10}\times\frac{x^2-4}{x^2-3x-10}\times\frac{x+1}{x^2+3x}$$

$$= \frac{(x-5)(x+5)}{(x+5)(x-2)}\times\frac{(x-2)(x+2)}{(x-5)(x+2)}\times\frac{x+1}{x(x+3)}$$

$$= \frac{x+1}{x(x+3)}$$

Example 8

$$\frac{x^2-3x+2}{(x-3)}\div\frac{x^2-1}{2x^2-6x}$$

$$= \frac{x^2-3x+2}{(x-3)}\times\frac{2x^2-6x}{x^2-1}$$

$$= \frac{(x-1)(x-2)}{(x-3)}\times\frac{2x(x-3)}{(x-1)(x+1)}$$

$$= \frac{2x(x-2)}{x+1}$$

Example 9

Given that $x=\frac{1+y}{2y-1}$ **and** $y=\frac{1+2z}{1-z}$ **. Find** z **in terms of** x **only.**

Now in the first equation

$x=\frac{1+y}{2y-1}$ **, substituting** $y=\frac{1+2z}{1-z}$ **, we get**

$$x=\frac{1+\frac{1+2z}{1-z}}{2\left(\frac{1+2z}{1-z}\right)-1}$$

$$x=\frac{\frac{(1-z)+(1+2z)}{1-z}}{\frac{2(1+2z)-(1-z)}{1-z}}$$

$$x = \frac{\frac{2+z}{1-z}}{\frac{1+5z}{1-z}}$$

$$x = \frac{2+z}{1+5z}$$

$$x(1+5z) = 2+z$$
$$x+5xz = 2+z$$
$$z(5x-1) = 2-x$$
$$z = \frac{2-x}{5x-1}$$

Exercise 3.1
Simplify.

1. $\dfrac{x}{2x-6}+\dfrac{3}{6-2x}+\dfrac{x}{2}$

2. $\dfrac{6}{x^2+2x-8}+\dfrac{7}{10-3x-x^2}$

3. $\dfrac{3}{x^2+2x-15}-\dfrac{1}{x^2-x-6}-\dfrac{2}{x^2+7x+10}$

4. $\dfrac{2x}{x^2-2x-3}+\dfrac{1}{x^2-1}+\dfrac{x}{x^2-4x+3}$

5. $x-\dfrac{1}{1-x}-\dfrac{x^2-3x-2}{x^2-1}$

6. $\dfrac{1}{x^2-5x+6}-\dfrac{2}{x^2-4x+3}+\dfrac{1}{x^2-3x+2}$

7. $\dfrac{1}{2x-1}-\dfrac{2x}{4x^2-1}-\dfrac{1}{2x^2-3x+1}$

8. $\dfrac{a-2}{a^2-9a+20}-\dfrac{a+2}{a^2-a-12}$

9. $\dfrac{a-2}{a+2}+\dfrac{a+2}{a-2}-\dfrac{a^2+4}{a^2-4}$

10. $\dfrac{1}{x^2-1}-\dfrac{1}{2x^2-6x+4}+\dfrac{3}{2x^2-2x-4}$

11. $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-c)(b-a)}+\dfrac{bc}{(c-a)(c-b)}$

12. $\dfrac{x-a}{x-b}+\dfrac{x-b}{x-a}-\dfrac{(a-b)^2}{(x-a)(x-b)}$

13. $\dfrac{a^2+3a+2}{a^2-4a-12}\times\dfrac{a^2-7a+6}{a^2-4}$

14. $\dfrac{a^3+b^3}{a(a^2-b^2)}\times\dfrac{a+b}{a-b}\times\dfrac{a^2-ab}{(a+b)^2}$

15. $\dfrac{1}{a^2+ab+b^2}\times\dfrac{2a}{a^3+b^3}\times\dfrac{a^4+a^2b^2+b^4}{4a^2}$

16. $\left(\dfrac{a}{a-1}-\dfrac{a+1}{a}\right)\div\left(\dfrac{a}{a+1}-\dfrac{a-1}{a}\right)$

17. $\left(2-\dfrac{y^2+z^2-x^2}{yz}\right)\div\left(2+\dfrac{x^2+y^2-z^2}{xy}\right)$

18. $\left(\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}\right)\div\left(\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}\right)$

19. If $y=x+\dfrac{1}{x}$ and $z=y-\dfrac{1}{y}$, find z in terms of x.

20. If $y=\dfrac{1-t}{1+t}$, express $\dfrac{1-y^2}{1+y^2}$ in terms of t.

21. If $x=\dfrac{1+a}{1-a}$ and $y=\dfrac{1-a}{1+a}$ find $\dfrac{x-y}{1+xy}$ in terms of a.

22. If $a=\dfrac{2b+1}{b-1}$, $b=\dfrac{c+1}{2c-1}$, express $\dfrac{2q+1}{q-1}$ in terms of x only.

4. Equations

Equations involving one variable.
We will consider the various methods of solving an equation in this chapter.

(a) Linear Equations.

Example 1.

$$\frac{3x+2}{x-1}-\frac{2(x-2)}{x+2}=1$$

Multiplying both sides by L.C.M. $(x-1)(x+2)$

$$(3x+2)(x+2)-2(x-2)(x-1)=(x-1)(x+2)$$

$$(3x^2+8x+4)-2(x^2-3x+2) \quad = x^2+x-2$$

$$x^2+14x=x^2+x-2$$

$$13x=-2$$

$$x=-\frac{2}{13}$$

(b) Quadratic Equation

The most general form of a quadratic equation is $ax^2+bx+c=0$ **, where a, b, c are real numbers and** $a \neq 0$.

Solution of the quadratic equation $ax^2+bx+c=0 \cdot a \neq 0$ **by the method of completion of squares.**

$$ax^2+bx+c=0; \qquad a \neq 0$$

$$x^2+\frac{b}{a}x+\frac{c}{a}=0 \quad \textbf{(dividing both sides by } a\textbf{)}$$

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

Adding both sides $\left(\frac{b}{2a}\right)^2$ **, we get**

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{-c}{a}+\left(\frac{b}{2a}\right)^2$$

$$\left(x+\frac{b}{2a}\right)^2 \quad =\frac{b^2-4ac}{4a^2}$$

$$x+\frac{b}{2a} \quad =\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x \quad = \quad \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-b+\sqrt{b^2-4ac}}{2a}, \quad \frac{-b-\sqrt{b^2-4ac}}{2a},$$

Thus the roots of the equation are

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a},$$

Example 2

Solve the following equations.

(a) $4x^2-4x-3=0$

(b) $3x^2-5x-1=0$

(a) $4x^2-4x-3=0$

This equation can be solved by factorising $4x^2-4x-3$

$$4x^2-4x-3=0$$

$$4x^2-6x+2x-3=0$$

$$2x(2x-3)+1(2x-3)=0$$

$$(2x-3)(2x+1)=0$$

$$2x-3=0 \quad or \quad 2x+1=0$$

$$x=\frac{3}{2} \quad or \quad x=-\frac{1}{2}$$

(b) $3x^2-5x-1=0$

(Method of completion of squares)

$$3x^2-5x-1=0$$

$$3x^2-5x=1$$

$$x^2-\frac{5}{3}x=\frac{1}{3}$$

$$x^2-\frac{5}{3}x+\left(\frac{-5}{6}\right)^2=\frac{1}{3}+\left(\frac{-5}{6}\right)^2$$

$$\left(x-\frac{5}{6}\right)^2=\frac{1}{3}+\frac{25}{36}=\frac{37}{36}$$

$$x-\frac{5}{6}=\pm\frac{\sqrt{37}}{6}$$

$$x=\frac{5+\sqrt{37}}{6} \quad or \quad \frac{5-\sqrt{37}}{6}$$

(c) Equations reducible to quadratic equations.

Example 3

Solve: $(x^2+3x)^2-5(x^2+3x)-6=0$

Put $y=x^2+3x$

$$y^2-5y-6=0$$
$$(y-6)(y+1)=0$$
$$y=6 \quad or \quad y=-1$$

$$x^2+3x=6 \qquad\qquad x^2+3x=-1$$
$$x^2+3x-6=0 \qquad\qquad x^2+3x+1=0$$
$$x=\frac{-3\pm\sqrt{9+24}}{2} \quad or \quad x=\frac{-3\pm\sqrt{9-4}}{2}$$
$$x=\frac{-3\pm\sqrt{33}}{2}, \qquad\qquad x=\frac{-3\pm\sqrt{5}}{2}$$

Roots of the equation are

$$\frac{-3+\sqrt{33}}{2},\quad \frac{-3-\sqrt{33}}{2},\quad \frac{-3+\sqrt{5}}{2},\quad \frac{-3-\sqrt{5}}{2}$$

Example 4.

$$\frac{4x+5}{x+5}+\frac{x+5}{4x+5}=\frac{10}{3}$$

Let $y=\dfrac{4x+5}{x+5}$

The given equation becomes

$$y+\frac{1}{y}=\frac{10}{3}$$
$$3y^2-10y+3=0$$
$$(3y-1)(y-3)=0$$
$$y=\frac{1}{3} \quad or \quad y=3$$

$$\frac{4x+5}{x+5}=\frac{1}{3} \quad or \quad \frac{4x+5}{x+5}=3$$

$$12x+15=x+5 \qquad\qquad 4x+5=3(x+5)$$
$$11x=-10 \qquad\qquad 4x+5=3x+15$$
$$x=\frac{-10}{11} \qquad\qquad x=10$$
$$x=\frac{-10}{11},\ 10$$

Example 5

(a) **Equations of the form** $a\left(x^2+\frac{1}{x^2}\right)+b\left(x+\frac{1}{x}\right)+c=0$

Solve: $6\left(x^2+\frac{1}{x^2}\right)+35\left(x+\frac{1}{x}\right)+62=0$

$$6\left(x^2+\frac{1}{x^2}\right)+35\left(x+\frac{1}{x}\right)+62=0$$

Let

$$x+\frac{1}{x}=y$$
$$\left(x+\frac{1}{x}\right)^2=y^2$$
$$x^2+\frac{1}{x^2}=y^2-2$$

The equation becomes

$$6(y^2-2)+35y+62=0$$
$$6y^2+35y+50=0$$
$$(2y+5)(3y+10)=0$$
$$y=-\frac{5}{2} \quad or \quad y=-\frac{10}{3}$$
$$x+\frac{1}{x}=-\frac{5}{2} \qquad or \qquad x+\frac{1}{x}=-\frac{10}{3}$$
$$2x^2+5x+2=0 \qquad 3x^2+10x+3=0$$
$$(2x+1)(x+2)=0 \qquad (3x+1)(x+3)=0$$
$$x=-\frac{1}{2} \text{ or} -2 \quad \text{or} \quad x=-\frac{1}{3} \text{ or } -3$$

Hence the roots of the equation are $-\frac{1}{2},\ -2,\ -\frac{1}{3},\ -3$

Example 6.

$$2\left(x+\frac{1}{x}\right)^2-3\left(x-\frac{1}{x}\right)=8$$

Let

$$y = x-\frac{1}{x}$$

$$y^2=\left(x-\frac{1}{x}\right)^2 = x^2-2+\frac{1}{x^2}$$

$$y^2+4=x^2+2+\frac{1}{x^2} = \left(x+\frac{1}{x}\right)^2$$

The given equation becomes

$$2(y^2+4)-3y=8$$

$$2y^2-3y = 0$$

$$y(2y-3) = 0$$

$$y=0 \quad or \quad y=\frac{3}{2}$$

$$x-\frac{1}{x}=0 \quad or \quad x-\frac{1}{x}=\frac{3}{2}$$

$$x^2-1=0 \quad or \quad 2x^2-3x-2=0$$

$$(x-1)(x+1)=0 \qquad (2x+1)(x-2)$$

$$x=1 \ or -1 \qquad x=-\frac{1}{2} \ or \ 2$$

Hence the solution set is $\left\{-1,\ -\frac{1}{2},\ 1,\ 2\right\}$

Example 7.

Solve the following equations.

(a) $2^{2x}-3.2^{x+2}+32=0$

(b) $3^{x-2}+3^{3-x}=4$

(a) $2^{2x}-3.2^{x+2}+32=0$

$$\left(2^x\right)^2-3\times 2^2\times 2^x+32=0$$

Let $y=2^x$

Then the given equation becomes

$$y^2 - 12y + 32 = 0$$
$$(y-8)(y-4) = 0$$
$$y = 8 \quad \text{or} \quad y = 4$$
$$2^x = 8 \quad \text{or} \quad 2^x = 4$$
$$2^x = 2^3 \quad \text{or} \quad 2^x = 2^2$$
$$x = 3 \quad \text{or} \quad x = 2$$

Hence roots are 3, 2

(b) $3^{x-2} + 3^{3-x} = 4$

$$3^x \times \frac{1}{3^2} + 3^3 \times \frac{1}{3^x} = 4$$

Let $y = 3^x$

The given equation becomes

$$\frac{y}{9} + \frac{27}{y} = 4$$
$$y^2 - 36y + 27 \times 9 = 0$$
$$(y-27)(y-9) = 0$$
$$y = 27 \quad or \quad y = 9$$
$$3^x = 27 \quad or \quad 3^x = 9$$
$$3^x = 3^3 \quad or \quad 3^x = 3^2$$
$$x = 3 \quad or \quad x = 2$$

x = 3, 2 are the solutions of the equations.

Example 8.

Solve the equation.

$$(x+1)(2x+1)(2x-7)(x-3) = 45$$

$$(x+1)(2x+1)(2x-7)(x-3) = 45$$
$$\left[(x+1)(2x-7)\right]\left[(2x+1)(x-3)\right] = 45$$
$$(2x^2-5x-7)(2x^2-5x-3) = 45$$

Let $y = 2x^2 - 5x$

The equation becomes

$$(y-7)(y-3)=45$$
$$y^2-10y+21=45$$
$$y^2-10y-24=0$$
$$(y+2)(y-12)=0$$
$$y+2=0 \;\; or \;\; y-12=0$$

$$2x^2-5x+2=0 \qquad\qquad 2x^2-5x-12=0$$
$$(2x-1)(x-2)=0 \qquad\qquad (2x+3)(x-4)=0$$
$$x=\frac{1}{2} \;\; or \;\; 2 \qquad\qquad x=-\frac{2}{3} \;\; or \;\; 4$$

The solution set is $\left\{\frac{1}{2},\ 2,\ -\frac{3}{2},\ 4\right\}$

Example 9

Solve : $\sqrt{4x-3}+\sqrt{2x+3}=6 \quad \left(x\geq\frac{3}{4}\right)$

The equation is valid only if

$$4x-3\geq 0 \quad and \quad 2x+3\geq 0$$
$$x\geq\frac{3}{4} \qquad and \qquad x\geq-\frac{3}{2}$$

Since the both conditions should be satisfied, the required condition for the values of x **is** $x\geq\frac{3}{4}$

$$\sqrt{4x-3}+\sqrt{2x+3}=6;\ \left(x\geq\frac{3}{4}\right)$$

Squaring both sides,

$$(4x-3)+2\sqrt{(4x-3)(2x+3)}+(2x+3)=36$$
$$6x-36=-2\sqrt{(4x-3)(2x+3)}$$
$$6x-36=-2\sqrt{(4x-3)(2x+3)}$$
$$3(x-6)=-\sqrt{(4x-3)(2x+3)}$$
$$9(x-6)^2=(4x-3)(2x+3)$$
$$9(x^2-12x+36)=8x^2+6x-9$$
$$x^2-114x+333=0$$
$$(x-111)(x-3)=0$$
$$x=3 \;\; or \;\; 111$$

Both 3 and 111 satisfy the condition $x \geq \frac{3}{4}$ **.**

Now we will varify the solution.

When $x = 3$**,**

$$\text{L.H.S.} = \sqrt{4x-3} + \sqrt{2x+3}$$
$$= \sqrt{9} + \sqrt{9} = 3 + 3 = 6 + \text{R.H.S., which is true}$$

When $x = 111$**,**

$$\text{L.H.S.} = \sqrt{4x-3} + \sqrt{2x+3}$$
$$= \sqrt{4 \times 111 - 3} + \sqrt{2 \times 111 - 3}$$
$$= \sqrt{441} + \sqrt{225}$$
$$= 21 + 15 = 36 \neq \text{R.H.S.}$$

Hence 3 is the only solution of the given equation.

Exercise 4(a)

Solve the following equations.

1 $3-2(2x+1)=7$

2 $\frac{x+9}{2}-\frac{2x-3}{2}=\frac{3x+4}{4}$

3 $\frac{x+3}{4}-\frac{x-3}{5}=2$

4 $\frac{2x}{15}-\frac{x-6}{12}-\frac{3x}{20}=\frac{3}{2}$

5 $6-\frac{4(x-3)}{3}=\frac{x-2}{5}$

6 $\frac{4-3x}{8}+2=\frac{x-5}{4}-x$

7. $\frac{3x-11}{x-4}-\frac{x+7}{x+4}=2$

8 $(x+1)(2x-1)+(x-3)(2x+1)=(2x+3)^2$

9 $\frac{5}{x-2}-\frac{3}{x+2}=\frac{2}{x+4}$

10. $\frac{3x-2}{4}-\frac{x-3}{5}=x+1$

Exercise 4(b)

Solve the following equations.

1 $3x^2-2x=0$

2 $(x+2)^2=1$

3 $(x-3)(x-5)=3$

4 $2x^2-5x-3=0$

5 $x^2-3x(3x-4)+8=0$

6 $5x(x+1)-x(2x+1)=4$

7. $x^2+(x+3)^2=15^2$

8 $\frac{3}{x-3}-\frac{4}{x-4}+\frac{5}{x-1}=0$

9 $\frac{x}{(x+2)(x-1)}+\frac{1}{(x+2)(2x-1)}-\frac{1}{(x-1)(2x-1)}=0$

10. $\frac{x-1}{(x-3)(x-2)}-\frac{x-2}{(x-3)(x-1)}=\frac{x+1}{(x-2)(x-1)}$

11 $\frac{2}{3(x+2)}-\frac{3}{(2x+7)}=\frac{1}{15}$

12. $\frac{14}{2x-1}-\frac{7}{x}=\frac{1}{3}$

Solve the following equations by the method of completion of squares.

13. $x^2-6x-5=0$

14. $2x^2+7x-5=0$

15. $2x^2-3x-7=0$

16. $\frac{x}{x+1}-\frac{x-1}{5}=0$

Exercise 4(c)

Solve the following equations.

1 $(x^2+5x+7)^2-4(x^2+5x+7)+3=0$

2 $(x^2-9x+15)\ (x^2-9x+20)=6$

3 $\left(x+\frac{2}{x}+4\right)\left(x+\frac{2}{x}-1\right)=6$

4 $\left(\frac{x}{x+1}\right)^2+5\left(\frac{x}{x+1}\right)+6=0$

5 $3\left[(x+7)^{\frac{1}{2}}+(x+7)^{-\frac{1}{2}}\right]=10$

6 $x+4\sqrt{x}=12$

7. $x+3\sqrt{5x}=50$

8 $x^{\frac{1}{2}}-x^{-\frac{1}{2}}=\frac{3}{2}$

9 $x^{\frac{1}{3}}+x^{\frac{2}{3}}=2$

10. $9x^{\frac{2}{3}}+4x^{-\frac{2}{3}}=37$

11 $\frac{x^2}{x^2+3x+2}+\frac{2(x^2+3x+2)}{x^2}=12\frac{1}{6}$

12. $\sqrt{\frac{x}{x-1}}+\sqrt{\frac{x-1}{x}}=2\frac{1}{6}$

13. $\frac{5}{x^2+6x+2}=\frac{3}{x^2+6x+1}-\frac{4}{x^2+6x+8}$

14. $\left(x-\frac{1}{x}\right)^2+7\left(x-\frac{1}{x}\right)=12\frac{3}{4}$

15. $2\left(x+\frac{1}{x}\right)^2-7\left(x+\frac{1}{x}\right)+6=0$

16. $9\left(x^2+\frac{1}{x^2}\right)-27\left(x+\frac{1}{x}\right)+8=0$

17. $\left(x^2+\frac{1}{x^2}\right)-5\left(x+\frac{1}{x}\right)+4=0$

18. $3\left(x^2+\frac{1}{x^2}\right)-16\left(x+\frac{1}{x}\right)+26=0$

19. $2\left(x^2+\frac{1}{x^2}\right)-9\left(x-\frac{1}{x}\right)+14=0$

20. $8\left(x^2+\frac{1}{x^2}\right)-42\left(x-\frac{1}{x}\right)+29=0$

21. $\left(x+\frac{1}{x}\right)^2-\frac{3}{2}\left(x-\frac{1}{x}\right)=4$

22. $3^{x+2}+3^{-x}=10$

23. $5^{x+1}+5^{1-x}=5^2+5^0$

24. $4^{1+x}+4^{1-x}=10$

25. $\sqrt{x+2}+\sqrt{x+9}=7$

26. $2\sqrt{x+1}-3\sqrt{2x-5}=\sqrt{x-2}$

27. $\sqrt{3x-5}-\sqrt{2x-5}=1$

28. $2^{2x+3}=65(2^x-1)+57$

29. $(x-6)(x-5)(x+1)(x+2)=144$

30. $\left(x-\frac{1}{x}\right)^2+2\left(x+\frac{1}{x}\right)=\frac{29}{4}$

31. $(x+1)(x+2)(x+3)(x+4)+1=0$

2. Equations in two variables

(a) Both equations are linear in two variables x and y.

These equations can be written in the form $ax + by = m,\ cx + dy = n$

Example 1

Solve: $4x + 3y = 17$
$5x - 2y = 4$

$4x + 3y = 17$ ———— (1)
$5x - 2y = 4$ ———— (2)
(1)x2, $8x + 6y = 34$ ———— (3)
(2)x3, $15x - 6y = 12$ ———— (4)
(3)+(4) $23x = 46$

$$x = \frac{46}{23} = 2$$

Substituting $x = 2$ in the equation (1)

$8 + 3y = 17$
$3y = 9$
$y = 3$
$x = 2,\ y = 3$

(b) One equation linear the other non-linear.

Example 2

Solve:
$2x - 3y = 1$
$2x^2 + 3x - 3y^2 = 38$

$2x - 3y = 1$ ———— (1)
$2x^2 + 3x - 3y^2 = 38$ ———— (2)

From the first equation $y = \frac{2x-1}{3}$

Substituting in the second equation.

$$2x^2 + 3x - 3y^2 = 38$$
$$2x^2 + 3x - 3\left(\frac{2x-1}{3}\right)^2 = 38$$
$$6x^2 + 9x - (4x^2 - 4x + 1) = 114$$
$$2x^2 + 13x - 115 = 0$$
$$(2x+23)(x-5) = 0$$

$x = 5 \qquad or \qquad x = -\frac{23}{2}$

If $x = 5 \qquad$ If $x = -\frac{23}{2}$

$y = \frac{2\times 5 - 1}{3} = 3 \qquad y = \frac{-23-1}{3} = -8$

$\left.\begin{array}{l} x = 5 \\ y = 3 \end{array}\right\} \qquad \left.\begin{array}{l} x = -\frac{23}{2} \\ y = -8 \end{array}\right\}$

(c) **Both equations, homogeneous expressions in *x* and *y* equal to a constant.**

Example 3

Solve:

$x^2 - xy = 6$ ———— (1)

$x^2 + y^2 = 61$ ———— (2)

(1)x 61, $\quad 61(x^2 - xy) = 61 \times 6$

(2)x 6, $\quad 6(x^2 + y^2) = 6 \times 61$

$61(x^2 - xy) = 6(x^2 + y^2)$

$55x^2 - 61xy - 6y^2 = 0$

$(11x + y)(5x - 6y) = 0$

$y = -11x \quad$ or $\quad y = \frac{5x}{6}$

When $y = -11x$

Equation (1) becomes

$12x^2 = 6$

$x^2 = \frac{1}{2}$

$x = \pm\frac{1}{\sqrt{2}}$

When $y = \frac{5x}{6}$

Equation (1) becomes

$x^2 = 36$

$x = \pm 6$

$\left.\begin{array}{l} x = \frac{1}{\sqrt{2}} \\ y = \frac{-11}{\sqrt{2}} \end{array}\right. \qquad \left.\begin{array}{l} x = -\frac{1}{\sqrt{2}} \\ y = \frac{11}{\sqrt{2}} \end{array}\right\} \qquad \left.\begin{array}{l} x = 6 \\ y = 5 \end{array}\right\} \qquad \left.\begin{array}{l} x = -6 \\ y = -5 \end{array}\right\}$

(d) One equation is homogeneous in the two variables x and y.

Example 4

Solve:

$x^2 + xy - 2y^2 = 0$ ——————— (1)

$x^2 + 2xy + 3y^2 + 4x + 5y = 15$ ——— (2)

Equation, $x^2 + xy - 2y^2 = 0$ is homogeneous.

$(x + 2y)(x - y) = 0$

$x = -2y$ **or** $x = y$

Substituting $x = y$ **in (2), we get**

$6y^2 + 9y - 15 = 0$

$2y^2 + 3y - 5 = 0$

$(2y + 5)(y - 1) = 0$

or $y = 1$

Since $x = y$

$x = 1$ $\quad x = -\frac{5}{2}$

$y = 1$ $\quad y = -\frac{5}{2}$ $\quad y = -\frac{5}{2}$

Substituting $x = -2y$ **in the equation (2) gives,**

$$3y^2 - 3y - 15 = 0$$
$$y^2 - y - 5 = 0$$
$$y = \frac{1 \pm \sqrt{21}}{2}$$

Since $x = -2y$

$x = -(1+\sqrt{21})$ $\quad x = -(1-\sqrt{21}$

$y = \frac{1+\sqrt{21}}{2}$ $\quad y = \frac{1-\sqrt{21}}{2}$

3. Further examples (including equations in three variables)

Example 5

Solve:

$x(3y - 5) = 4$ ——— (1)

$y(2x + 7) = 27$ ——— (2)

From equation (1) $x = \frac{4}{3y-5}$

$$2x+7 = 2\times\frac{4}{3y-5}+7$$
$$= \frac{8}{3y-5}+7$$
$$= \frac{21y-27}{3y-5}$$

Substituting this in equation (2), it becomes

$$y\left(\frac{21y-27}{3y-5}\right) = 27$$
$$21y^2-27y = 27(3y-5)$$
$$21y^2-108y+135 = 0$$
$$7y^2-36y+45 = 0$$
$$(7y-15)(y-3) = 0$$
$$y = \frac{15}{7} \quad or \quad y = 3$$

$x, y \neq 0$

When $y = \frac{15}{7}$ **when** $y = 3$

$$x = \frac{4}{3\times\frac{15}{7}-5} = \frac{14}{5}$$

$$x = \frac{4}{3\times 3-5} = \frac{4}{4} = 1$$

$$\left.\begin{aligned} x &= \frac{14}{5} \\ y &= \frac{15}{7} \end{aligned}\right\}$$

$$\left.\begin{aligned} x &= 1 \\ y &= 3 \end{aligned}\right\}$$

Example 6

Solve the equations

$3x+5y = 29xy$ ———— (1)

$7x+4y = 37xy$ ———— (2)

If $x = 0$, **then** $y = 0$.

i.e., $x = 0$, $y = 0$ **satisfies the given equations.**

Let

Dividing both sides of equations by xy.

$$\frac{3x}{xy}+\frac{5y}{xy}=29 \text{ ——————— } (3)$$

$$\frac{7x}{xy}+\frac{4y}{xy}=37 \text{ ——————— } (4)$$

$$\frac{3}{y}+\frac{5}{x}=29$$

$$\frac{7}{y}+\frac{4}{x}=37$$

4 x (3) - 5 x (4) gives

$$\frac{12}{y}-\frac{35}{y}=116-185$$

$$-\frac{23}{y}=-69$$

$$y=\frac{1}{3}$$

Substituting $y=\frac{1}{3}$ **in equation (3), we get** $9+\frac{5}{x}=29$

$$x=\frac{1}{4}$$

Hence the solutions: $\left.\begin{matrix} x=0 \\ y=0 \end{matrix}\right\}$ $\left.\begin{matrix} x=\frac{1}{4} \\ y=\frac{1}{3} \end{matrix}\right\}$

Example 7

Solve the equations

$a+4b+4c = 7$ ——————— **(1)**

$3a+2b+2c = 6$ ——————— **(2)**

$9a+6b+2c = 14$ ——————— **(3)**

2 x (2) - (1) gives $5a = 12-7=5$

$a = 1$

3 x (2) - (3) gives $4c = 4$

$c = 1$

Substituting $a=1$ **and** $c=1$ **in equation (1)**

$1+4b+4 = 7$

$4b = 2$, $b=\frac{1}{2}$

Hence the solution is $\left.\begin{aligned} a &= 1 \\ b &= \frac{1}{2} \\ c &= 1 \end{aligned}\right\}$

Example 8

Solve the equations.

$x + y = 1$ ———— (1)
$y + z = 2$ ———— (2)
$z + x = 5$ ———— (3)

(1)+(2)+(3) gives,

$$2(x + y + z) = 8$$
$$x + y + z \quad = 4 \text{ ———— (4)}$$

Substituging $x + y = 1 \ \text{ in}(4), \ z = 3$

Substituting $y + z = 2 \ \text{ in}(4), \ \text{x} = 2$

Substituting $x + y = 5 \ \text{ in}(4), \ \text{y} = -1$

Hence $x = 2, y = -1, \ z = 3$

Example 9 $\times$

Solve the equations;

$ab = 3$ ———— (1)
$bc = 6$ ———— (2)
$ac = 2$ ———— (3)

(1) (2) (3) gives,

$$(ab)\times(bc)\times(ac) = 3\times 6\times 2$$
$$a^2b^2c^2 = 36$$
$$abc \quad = \pm 6$$

Let $abc = 6$

From equation (1), $c = 2$

From equation (2), $a = 1$

From equation (3), $b = 3$

Let $abc = -6$

From equation (1), $c = -2$

From equation (2), $a = -1$

From equation (3), $b = -3$

Hence the solutions are

$\left.\begin{aligned} a &= 1 \\ b &= 3 \\ c &= 2 \end{aligned}\right\}$ $\qquad$ $\left.\begin{aligned} a &= -1 \\ b &= -3 \\ c &= -2 \end{aligned}\right\}$

Exercise 4(d)

Solve.

1. $x+2y=4$
   $3x+5y=9$

2. $3x-2y=7$
   $2x-5y=12$

3. $5x-3y=18$
   $3x=11+2y$

4. $53x+47y=59$
   $47x+53y=41$

5. $\frac{x-1}{2}=\frac{y+2}{3}=\frac{2x+2y}{9}$

6. $\frac{2}{x}+\frac{1}{y}=18$
   $\frac{1}{x}-\frac{2}{y}=-1$

7. $\frac{2}{x}+\frac{3}{y}=-5$
   $\frac{3}{x}-\frac{5}{y}=21$

8. $\frac{3}{x}-\frac{2}{y}=2$
   $\frac{9}{x}-\frac{4}{y}=1$

9. $5x-\frac{2}{y}=9$
   $2x-\frac{5}{y}=12$

10. $4x+\frac{5}{y}=3$
    $3x-\frac{4}{y}=10$

11. $ax-by=bx-ay=a^2-b^2$

12. $\frac{3}{x+1}+\frac{2}{y-4}=2$
    $\frac{4}{x+1}-\frac{9}{y-4}=5$

13. $\frac{x-2}{y}=\frac{1}{3}$
    $\frac{x}{y+1}=\frac{1}{2}$

14. $\frac{x+y}{xy}=2$
    $\frac{x-y}{xy}=6$

15. $(a+b)x+(a-b)y=2a$
    $(a-b)x+(a+b)y=2b$

Exercise 4(e)

Solve.

1. $y-2x=1$
   $y^2=2x^2+x$

2. $x-2y=1$
   $x^2-2xy+2y^2=25$

3. $2x+3y=5$
   $x^2+2xy=10+y$

4. $x+y=4$
   $x^2-y=8$

5. $3x+2y=25$
$xy=4$

6. $2y-3x=2$
$4y^2-4xy-18x^2=5$

7. $x^2-y^2=7$
$x=y^2-5$

8. $4x^2-3y^2=13$
$5x^2+2y=18$

9. $x^2+xy-y^2+6x-1=0$
$3x^2+5xy-2y^2=0$

10. $x^2+xy=2y^2$
$x^2+2xy+3y^2+4x+5y=15$

11. $x^2-2xy-y^2=14$
$2x^2+3xy+y^2=-2$

12. $x^2-xy+3y^2=15$
$3x^2-2y^2=-5$

13. $(x-2)(y-1)=3$
$(x+2)(2y-5)=15$

14. $\frac{x}{3}+\frac{3}{y}=\frac{x}{4}-\frac{4}{y}=1$

15. $x(y+3)=4$
$3y(x-4)=5$

16. $x-2y+3z=17$
$2x+y+5z=17$
$3x-4y-2z=1$

17. $2x+3y-4z=10$
$4x-5y+3z=2$
$2y+z=8$

18. $x+3y-2z=19$
$3x-y-z=7$
$-2x+5y+z=2$

19. $4x+3y-2z=11$
$3x-7y+3z=10$
$9x-8y+5z=8$

20. **Solve the equations $xy=1$, $yz=9$, $zx=16$ and deduce the solutions of equations.**
$(y+z)(z+x)=1$, $(z+x)(x+y)=9$, $(x+y)(y+z)=16$

21. **Solve the following equations.**
$(y-2)(z-1)=4$
$(z-1)(x+1)=20$
$(x+1)(y-2)=5$

22. **Solve the equations.**
$x(y+z)=33$, $y(z+x)=35$, $z(x+y)=14$

23. $y(z-x)=3$, $x(y+z)=32$, $x+y+z=12$

5. Indices and Logarithms

Laws of indices:

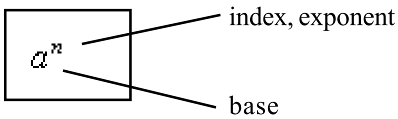
a, b are real numbers. m and n are rational numbers.

$$a^m \times a^n = a^{m+n}$$
$$a^m \div a^n = a^{m-n}$$
$$(ab)^m = a^m b^m$$
$$(a^m)^n = a^{mn}$$

When $a \neq 0$, and is a rational number.

$$a^{-n} = \frac{1}{a^n} \quad \text{and} \quad a^0 = 1$$

In a^n a is called the base and n the index or exponent.



An equation in which the variable is an exponent is called exponential equation.
For example $2^x = 32$ is an exponential equation.

Example 1

Find the values of the following when $x = 9$ and $y = 16$.

(a) $x^{\frac{1}{2}}.y^{\frac{3}{4}}$ (b) $\left(\frac{6x}{y}\right)^{\frac{1}{3}}$ (c) $(4xy)^{-\frac{1}{2}}$ (d) $(x+y)^{-\frac{1}{2}}$

(a) $$x^{\frac{1}{2}}.y^{\frac{3}{4}} = 9^{\frac{1}{2}} \times 16^{\frac{3}{4}} = (3^2)^{\frac{1}{2}} \times (2^4)^{\frac{3}{4}} = 3 \times 2^3 = 3 \times 8 = 24$$

(b) $$\left(\frac{6x}{y}\right)^{\frac{1}{3}} = \left(\frac{6 \times 9}{16}\right)^{\frac{1}{3}} = \left(\frac{27}{8}\right)^{\frac{1}{3}} = \left[\left(\frac{3}{2}\right)^3\right]^{\frac{1}{3}} = \frac{3}{2}$$

(c) $$(4xy)^{-\frac{1}{2}} = (4 \times 9 \times 16)^{-\frac{1}{2}} = \left[(2 \times 3 \times 4)^2\right]^{-\frac{1}{2}} = (2 \times 3 \times 4)^{-1} = 24^{-1} = \frac{1}{24}$$

(d) $$(x+y)^{-\frac{1}{2}} = (9+16)^{-\frac{1}{2}} = 25^{-\frac{1}{2}} = (5^2)^{-\frac{1}{2}} = 5^{-1} = \frac{1}{5}$$

Example 2

Solve : (a) $2^x = 10^3 \times 5^{-x}$

(b) $16^{x-1} = \frac{1}{8}$

(a)

$$\begin{aligned} 2^x &= 10^3 \times 5^{-x} \\ 2^x &= 10^3 \times \frac{1}{5^x} \\ 2^x \times 5^x &= 10^3 \\ 10^x &= 10^3 \\ x &= 3 \end{aligned}$$

(b)

$$\begin{aligned} 16^{x-1} &= \frac{1}{8} \\ (2^4)^{x-1} &= \frac{1}{2^3} \\ 2^{4x-4} &= 2^{-3} \\ 4x-4 &= -3 \\ 4x &= 1 \\ x &= \frac{1}{4} \end{aligned}$$

Logarithm

Consider $y = 3^x$ **. It will be observed that** y **must be positive for all real values of** x.

When x **=2,** y **= 9**
x **=3,** y **= 27**
x **=0,** y **= 1**
x **=-4,** y **=** $\frac{1}{81}$

In $3^x = y$ **, 3 is called base and** x **is index. The logarithm of the number** $y(>0)$ **to the base 3 is** x**.**

ie; $3^x = y \Leftrightarrow \log_3 y = x$

In general, if $a^x = y, (a > 0, y > 0)$ **then** x **is called the logarithm of** y **to the base** a **and is written as** $\log_a y = x$

$$a^x = y \Leftrightarrow \log_a y = x, \; a, y > 0, \; a \neq 1$$

For example:

$$\begin{aligned} &2^5 = 32 \Leftrightarrow \log_2 32 = 5 \\ &10^3 = 1000 \Leftrightarrow \log_{10} 1000 = 3 \\ &3^{-4} = \frac{1}{81} \Leftrightarrow \log_3 \frac{1}{81} = -4 \\ &\left(\frac{1}{2}\right)^5 = \frac{1}{32} \Leftrightarrow \log_{\frac{1}{2}} \frac{1}{32} = 5 \\ &a^1 = a \Leftrightarrow \log_a a = 1 \end{aligned}$$

Some fundamental properties of logarithms.

m, n **and** a **are positive numbers and** $a \neq 1$.
(a) $\log_a mn = \log_a m + \log_a n$
(b) $\log_a \frac{m}{n} = \log_a m - \log_a n$
(c) $\log_a m^p = p \log_a m$ **where** p **is rational.**

Let $\log_a m = x$ **and** $\log_a n = y$

$$\log_a m = x \Leftrightarrow m = a^x$$
$$\log_a n = y \Leftrightarrow n = a^y$$

(a) $mn = a^x \times a^y = a^{x+y} \Leftrightarrow \log_a mn = x + y$

$\log_a mn = x + y = \log_a m + \log_a n$

(b) $\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y} \Leftrightarrow \log_a \left(\frac{m}{n}\right) = x - y$

$\log_a \frac{m}{n} = x - y = \log_a m - \log_a n$

(c) $m^p = \left(a^x\right)^p = a^{px}$

$\log_a m^p = px = p \log_a m$

Example 1
Find the values of the following

(a) $\log_{10} 5 - \log_{10} 16 + 2\log_{10} 2 + \log_{10} 8$

(b) $\log_{10} 54 - \log_{10} 15 + 2\log_{10} \frac{5}{3}$

(a) $\log_{10} 5 - \log_{10} 16 + 2\log_{10} 2 + \log_{10} 8$

$= \log_{10} 5 - \log_{10} 16 + \log_{10} 2^2 + \log_{10} 8$

$= \log_{10} \left(\frac{5 \times 2^2 \times 8}{16}\right)$

$= \log_{10} 10 = 1$

(b) $\log_{10} 54 - \log_{10} 15 + 2\log_{10} \frac{5}{3}$

$= \log_{10} 54 - \log_{10} 15 + \log_{10} \frac{5}{3}$

$= \log_{10} 54 - \log_{10} 15 + \log_{10} \left(\frac{5}{3}\right)^2$

$= \log_{10} \left(\frac{54 \times \left(\frac{5}{3}\right)^2}{15} \right)$

$= \log_{10} \left(\frac{54 \times 25}{9 \times 15} \right)$

$= \log_{10} 10 = 1$

Example 2

Solve: (a) $3\log x + \log 96 = 2\log 9 + \log 4$

(b) $4\log x + 6\log 3 = \log 625 + \log 9$

(a) $3\log x + \log 96 = 2\log 9 + \log 4$

$\log x^3 + \log 96 = \log 9^2 + \log 4$

$\log\left(x^3 \times 96\right) = \log\left(9^2 \times 4\right)$

$x^3 \times 96 = 9^2 \times 4$

$x^3 = \frac{9^2 \times 4}{96}$

$x^3 = \frac{27}{8} = \left(\frac{3}{2}\right)^3$

$x = \frac{3}{2}$

(b) $4\log x + 6\log 3 = \log 625 + \log 9$

$\log x^4 + \log 3^6 = \log 625 + \log 9$

$\log\left(x^4 \times 3^6\right) = \log\left(625 \times 9\right)$

$x^4 \times 3^6 = 625 \times 9$

$x^4 = \frac{625 \times 9}{3^6} = \left(\frac{5}{3}\right)^4$

$x = \frac{5}{3}$

Example 3

Given that $\log_{10} 2 = 0.3010$ **and** $\log_{10} 3 = 0.4771$ **, find the values of (a)** $\log_{10} 18$ **(b)** $\log_{10} 15$ **(c)** $\log_{10} 0.012$

(a) $\log_{10} 18 = \log_{10}(2\times 3^2)$

$$= \log_{10} 2 + \log_{10} 3^2$$
$$= \log_{10} 2 + 2\log_{10} 3$$
$$= 0.3010 + 2\times 0.4771$$
$$= 0.3010 + 0.9542 = 1.2552$$

(b) $\log_{10} 15 = \log_{10}(5\times 3)$

$$= \log_{10} 5 + \log_{10} 3$$
$$= \log_{10} \frac{10}{2} + \log_{10} 3$$
$$= \log_{10} 10 - \log_{10} 2 + \log_{10} 3$$
$$= 1 - 0.3010 + 0.4771$$
$$= 1.1761$$

(c) $\log_{10} 0.012 = \log_{10} \frac{12}{1000}$

$$= \log_{10} 12 - \log_{10} 1000$$
$$= \log_{10}\left(2^2 \times 3\right) - \log_{10} 10^3$$
$$= \log_{10} 2^2 + \log_{10} 3 - 3\log_{10} 10$$
$$= 2\log_{10} 2 + \log_{10} 3 - 3\log_{10} 10$$
$$= 2\times 0.3010 + 0.4771 - 3$$
$$= 1.0791 - 3$$
$$= 1 + 0.0791 - 3$$
$$= -2 + 0.0791 \quad = \bar{2}.0791$$

Example 4

Find the value of

(a) $\log_{\sqrt{3}} \frac{1}{243}$ **(b)** $\log_{2\sqrt{2}} 16$

Let $\log_{\sqrt{3}} \frac{1}{243} = x$

$\left(\sqrt{3}\right)^{x} = 243$

$3^{\frac{1}{2}x} = 3^{5}$

$\frac{1}{2}x = 5$

$x = 10$

Let $\log_{2\sqrt{2}} 16 = y$

$\left(2\sqrt{2}\right)^{y} = 16$

$\left(2\times 2^{\frac{1}{2}}\right)^{y} = 2^{4}$

$2^{\frac{3}{2}y} = 2^{4}$

$\frac{3}{2}y = 4$

$y = \frac{8}{3}$

Exercise 5

1. If $x = 27$ **and** $y = 4$ **find the values of**

(a) $\left(x^{\frac{2}{3}}y\right)^{\frac{1}{2}}$ **(b)** $(2xy)^{-\frac{1}{3}}$ **(c)** $\left(\frac{12y}{x}\right)^{\frac{1}{2}}$ **(d)** $\left(x^{\frac{2}{3}}+y^{2}\right)^{-\frac{1}{2}}$

2 Find the values of the following:

(a) $\left(25^{\frac{1}{2}}\times 16^{\frac{1}{4}}\right)^{-2}$ **(b)** $\left(\frac{64^{\frac{1}{6}}+27^{\frac{2}{3}}}{110}\right)^{2}$ **(c)** $\left(\frac{81}{24}\right)^{\frac{1}{3}}$

3 If $x = 81$, $y = 16$ **and** $z = 25$, **find the values of**

(a) $(xy)^{\frac{3}{4}}$ **(b)** $x^{\frac{1}{2}}+y^{\frac{1}{2}}$ **(c)** $\left[\frac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{z^{-\frac{1}{2}}}\right]^{\frac{1}{2}}$

4 Simplify.

$$\left(\frac{4}{9}\right)^{\frac{3}{2}}\times\frac{1}{3^{-4}}\times\left(\frac{27}{8}\right)^{\frac{1}{3}}$$

5 Solve:

(a) $3^{x+1} = 243$ **(b)** $16^{x-1} = \frac{1}{8}$

(c) $4^{3x-1} = \left(\frac{1}{2}\right)^{x-1}$ **(d)** $27^{x-3} = 3\times 9^{x-2}$

(e) $3^{x^{2}} = 9^{x+4}$ **(f)** $9^{x} - 4\times 3^{x} + 3 = 0$

6 Evaluate each of the following:

(i) $\log_{3} 81$ **(ii)** $\log_{3\sqrt{2}} 324$ **(iii)** $\log_{2\sqrt{3}} 144$ **(iv)** $\log_{343} 7$

7. **Find the values of**

(a) $\log_{10}\frac{12}{5}+\log_{10}\frac{25}{21}-\log_{10}\frac{2}{7}$ (b) $\log_{10}\frac{3}{4}+\log_{10}\frac{10}{9}+\log_{10}12-2$

(c) $\frac{\log_{10}8}{\log_{10}4}$ (d) $3\log_{10}2+2\log_{10}5-\log_{10}2$

8. **Find the value of x in the following equations.**

(a) $5\log x-\log 729=6\log 2+11\log x$

(b) $4\log x+2\log 9=3\log 24-\log 54$

(c) $2\log x=\log 3+\log(2x-3)$

9. **Solve the equations.**

(a) $2^{2+2x}+3.2^{x}-1=0$

(b) $\log_{10}(x^2+1)-2\log_{10}x=1$

10. **Show that**

$$\log_{10}2+16\log_{10}\frac{16}{15}+12\log_{10}\frac{25}{24}+7\log_{10}\frac{81}{80}=1$$

11. **Prove that**

(a) $\log(ab^2)-\log(ac)+\log(bc^4)-3\log(bc)=0$ **the base being the same through out.**

(b) $\log(\log x^5)-\log(\log x^2)=\log\frac{5}{2}$

(c) $\log\left(\frac{a^2}{bc}\right)+\log\left(\frac{b^2}{ca}\right)+\log\left(\frac{c^2}{ab}\right)=0$

12. **If** $a^2+b^2=7ab$ **, show that** $\log\left(\frac{a+b}{3}\right)=\frac{1}{2}\log a+\frac{1}{2}\log b$

13. **If** $\log\left(\frac{x+y}{2}\right)=\frac{1}{2}\log x+\frac{1}{2}\log y$ **, prove that** $x=y$

14. **Prove that** $\log(1+2+3)=\log 1+\log 2+\log 3$

15. **If** x,y,z **are any consecutive three positive integers, prove that** $\log(1+xz)=2\log y$

16. **Prove that** $\log a+\log a^2+\ldots\ldots+\log a^{2n}=n(2n+1)\log a,$ **where** $a>0.$

17. **If** $\log(x+y)=\log x-\log y,$ **show that** $x(1-y)=y^2$

18. **If** $2^x.3^y = 3^x.4^x = 6,$ **, show that** $x^2 - 2y^2 = 2x - 3y$

19. **If** $\frac{\log x}{2} = \frac{\log y}{3} = \frac{\log z}{5}$, **prove that**

(i) $xy = z$ **and** **(ii)** $x^8 = y^2 z^2$

20. **If** $\frac{\log x}{1} = \frac{\log y}{3} = \frac{\log z}{5}$, **prove that** $x^5.y^3.z^{-2} = 1$

6. Ratio and Proportion

Proportion : **Equality of two ratios is called a proportion.** $\frac{a}{b}=\frac{c}{d}$ **is a proportion. This is written as** $a:b=c:d$

Here a,b,c,d **are called proportionals.**

Properties of proportions.

If $a:b=c:d$ **, then**

(1) $\frac{a+b}{b}=\frac{c+d}{d}$

(2) $\frac{a-b}{b}=\frac{c-d}{d}$

(3) $\frac{a-b}{a-b}=\frac{c+d}{c-d}$

Let $\frac{a}{b}=\frac{c}{d}=k$

$\Rightarrow a=kb$ **and** $c=kd$

(1) $\frac{a+b}{b}=\frac{kb+b}{b}=\frac{b(k+1)}{b}=k+1$

$\frac{c+d}{d}=\frac{kd+d}{d}=\frac{d(k+1)}{d}=k+1$

Hence $\frac{a+b}{b}=\frac{c+d}{d}$

(2) $\frac{a-b}{b}=\frac{kb-b}{b}=\frac{b(k-1)}{b}=k-1$

$\frac{c-d}{d}=\frac{kd-d}{d}=\frac{d(k-1)}{d}=k-1$

Hence $\frac{a-b}{b}=\frac{c-d}{d}$

(3) $\frac{a+b}{a-b}=\frac{kb+b}{kb-b}=\frac{(k+1)b}{(k-1)b}=\frac{k+1}{k-1}$

$\frac{c+d}{c-d}=\frac{kd+d}{kd-d}=\frac{(k+1)d}{(k-1)d}=\frac{k+1}{k-1}$

Hence $\frac{a+b}{a-b}=\frac{c+d}{c-d}$

If $\frac{a}{b}=\frac{c}{d}$, then each ratio is equal to $\frac{ma+nc}{mb+nd}$

i.e., If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{b}=\frac{c}{d}=\frac{ma+nc}{mb+nd}$

Let $\frac{a}{b}=\frac{c}{d}=k$

$a=kb$ and $c=kd$

$$\frac{ma+nc}{mb+nd}=\frac{kmb+knd}{mb+nd}=\frac{k(mb+nd)}{(mb+nd)}=k$$

Hence $\frac{a}{b}=\frac{c}{d}=\frac{ma+nc}{mb+nd}$

This is a very useful result in solving problems.

Example 1

If $\frac{4a+b}{2a+b}=7$, find the values of (a) $\frac{5a+b}{5a-b}$ (b) $\frac{b^2-a^2}{a^2+b^2}$

$$\frac{4a+b}{2a+b}=7$$

$$\begin{aligned}4a+b&=14a+7b\\10a&=-6b\\5a&=-3b\\a&=-\frac{3b}{5}\end{aligned}$$

(a) $\frac{5a+b}{5a-b}=\frac{-3b+b}{-3b-b}=\frac{-2b}{-4b}=\frac{1}{2}$

(b) $\frac{b^2-a^2}{b^2+a^2}=\frac{b^2-\frac{9b^2}{25}}{b^2+\frac{9b^2}{25}}=\frac{16}{34}=\frac{8}{17}$

Example 2
Solve the equations.

$$2x-3y=0$$
$$3x+4y=51$$

$$2x-3y=0, \qquad 2x=3y$$
$$\frac{x}{3}=\frac{y}{2}$$

Let $\frac{x}{3}=\frac{y}{2}=k$

Then $k=\frac{x}{3}=\frac{y}{2}=\frac{3x+4y}{3\times 3+4\times 2}=\frac{51}{17}=3$

$$\frac{x}{3}=\frac{y}{2}=3$$
$$\left.\begin{array}{l} x=9 \\ y=6 \end{array}\right\}$$

Example 3

If $\frac{x}{y}=\frac{a}{b}$, show that $\frac{2x+3y}{2x-3y}=\frac{2a+3b}{2a-3b}$

$\frac{x}{y}=\frac{a}{b}$ implies that $\frac{x}{a}=\frac{y}{b}$

Let $k=\frac{x}{a}=\frac{y}{b}$

Now $k=\frac{x}{a}=\frac{y}{b}=\frac{2x+3y}{2a+3b}$

$$k=\frac{x}{a}=\frac{y}{b}=\frac{2x-3y}{2a-3b}$$
$$\frac{2x+3y}{2a+3b}=\frac{2x-3y}{2a-3b}$$
$$\frac{2x+3y}{2x-3y}=\frac{2a+3b}{2a-3b}$$

Alternate Method

Let $\frac{x}{y}=\frac{a}{b}=k$

$x=ky$ and $a=kb$

$$\frac{2x+3y}{2x-3y}=\frac{2ky+3y}{2ky-3y}=\frac{2k+3}{2k-3}$$

$$\frac{2a+3b}{2a-3b}=\frac{2kb+3b}{2kb-3b}=\frac{2k+3}{2k-3}$$

Hence $\frac{2x+3y}{2x-3y}=\frac{2a+3b}{2a-3b}$

Example 4

If $(4a+b)(4c-7d)=(4a-7b)(4c+d)$

Show that $a:b=c:d$

$$(4a+b)(4c-7d)=(4a-7b)(4c+d)$$

$$\frac{4a+b}{4c+d}=\frac{4a-7b}{4c-7d}=k(say)$$

$$k=\frac{4a+b}{4c+d}=\frac{4a-7b}{4c-7d}=\frac{(4a+b)-(4a-7b)}{(4c+d)-(4c-7d)}=\frac{8b}{8d}=\frac{b}{d}$$

$$k=\frac{4a+b}{4c+d}=\frac{4a-7b}{4c-7d}$$

$$k=\frac{7(4a+b)}{7(4c+d)}=\frac{4a-7b}{4c-7d}=\frac{7(4a+b)+(4a-7b)}{7(4c+d)+(4c-7d)}=\frac{32a}{32c}=\frac{a}{c}$$

Her $\frac{a}{c}=\frac{b}{d}$

$$\Rightarrow \frac{a}{b}=\frac{c}{d}$$

Example 4 (Alternate Method)

If $(4a+7b)(4c-7d)=(4a-7b)(4c+7d)$

Show that $a:b=c:d$

Let $\frac{a}{b}=m$ **and** $\frac{c}{d}=n$

$a=mb$ **and** $c=nd$

Substituting $a=mb$ **and** $c=nd$ **in the equation**

$$(4a+7b)(4c-7d)=(4a-7b)(4c+7d)$$

We get, $(4mb+7b)(4nd-7d)=(4mb-7b)(4nd+7d)$

$$bd(4m+7)(4n-7) \quad = bd(4m-7)(4n+7)$$

$$b, d \neq 0$$

$$\therefore (4m+7)(4n-7) \quad = (4m-7)(4n+7)$$

$$16mn+28n-28m-49 = 16mn+28m-28n-49$$

$$28n-28m = 28m-28n$$

$$n-m = m-n$$

$$2m = 2n$$

$$m = n$$

Hence $a:b=c:d$

Example 5

If $x=\dfrac{6ab}{a+b}$, find the value of $\dfrac{x+3a}{x-3a}+\dfrac{x+3b}{x-3b}$

$$x=\frac{6ab}{a+b} \quad \Rightarrow \quad \frac{x}{2b}=\frac{3a}{a+b}$$

$$\frac{x}{2b}=\frac{3a}{a+b} \quad = \quad \frac{x+3a}{2b+(a+b)}=\frac{x-3a}{2b-(a+b)}$$

$$\frac{x+3a}{3b+a} \quad = \quad \frac{x-3a}{b-a}$$

$$\frac{x+3a}{x-3a} \quad = \quad \frac{3b+a}{b-a} \qquad \text{——————} \quad (1)$$

$$x=\frac{6ab}{a+b} \quad \Rightarrow \quad \frac{x}{2a}=\frac{3b}{a+b}$$

$$\frac{x}{2a}=\frac{3b}{a+b} \quad = \quad \frac{x+3b}{2a+(a+b)}=\frac{x-3b}{2a-(a+b)}$$

$$\frac{x+3b}{3a+b} \quad = \quad \frac{x-3b}{a-b}$$

$$\frac{x+3b}{x-3b} \quad = \quad \frac{3a+b}{a-b} \qquad \text{——————} \quad (2)$$

From (1) and (2)

$$\frac{x+3a}{x-3a}+\frac{x+3b}{x-3b} \quad = \quad \frac{3b+a}{b-a}+\frac{3a+b}{a-b}$$

$$= \quad \frac{(3b+a)-(3a+b)}{b-a}$$

$$= \quad \frac{2(b-a)}{b-a}$$

$$= \quad 2$$

Exercise 6

1.(a) If $a:b=5:3$ and $b:c=4:5$ find $a:b:c$

(b) If $x:y=3:4$ and $x:z=4:5$, find $x:y:z$

2 If $x:y=7:5$ find $5x-2y:3x+2y$

3 If $3x+5y:5x+12y=11:12$, find $x:y$

4 If, $5a^2-ab:2ab-b^2=6:1$, find $a:b$

5 If $a:b=c:d$, prove that

(a) $(2a+3b):(2c+3d)=(2a-3b):(2c-3d)$

(b) $(3a+5b):(3a-5b)=(3c+5d):(3c-5d)$

6 If $x=\dfrac{2ab}{a+b}$, find the value of $\dfrac{x+a}{x-a}+\dfrac{x+b}{x-b}$

7. If $x=\dfrac{10ab}{a+b}$, find the value of $\dfrac{x+5a}{x-5a}+\dfrac{x+5b}{x-5b}$

8 If $(2a+3b)(2c-3d)=(2a-3b)(2c+3d)$, show that $a:b=c:d$

9 If $(3a+6b-c-2d)(3a-6b+c-2d)=(3a+6b+c+2d)(3a-6b-c+2d)$, show that $a:b=c:d$

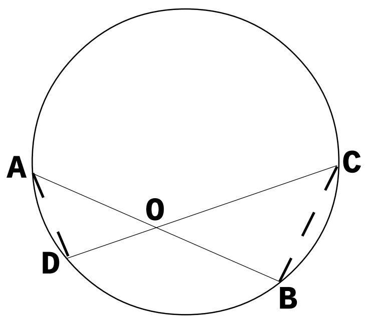
10. Solve the following equations using the properties of proportion.

(a) $\dfrac{x^2+1}{2x}=\dfrac{5}{4}$ (b) $\dfrac{x^3+3x}{3x^2+1}=\dfrac{341}{91}$

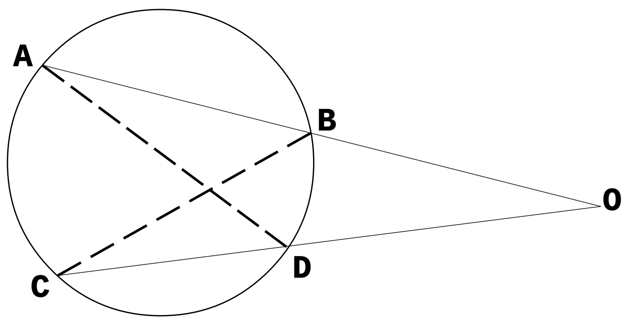
7.Rectangles in connection with circles

Theorem

If any two chords of a circle cut one another internally or externally, the rectangle obtained by the segments of one is equal to the rectangle obtained by the segments of the other.



(i)



(ii)

Given : Let the chords AB, CD cut one another at O, [internally in figure (i), externally in figure (ii)]

To prove : OA . OB = OC . OD

Construction : Join AD, BC.

Proof : In triangles AOD, COB.

$\angle$ OAD = $\angle$ OCB (Angles in the same segment)

$\angle$ AOD = $\angle$ COB (Vertically opposite angles)

Therefore, third angles in each are equal.

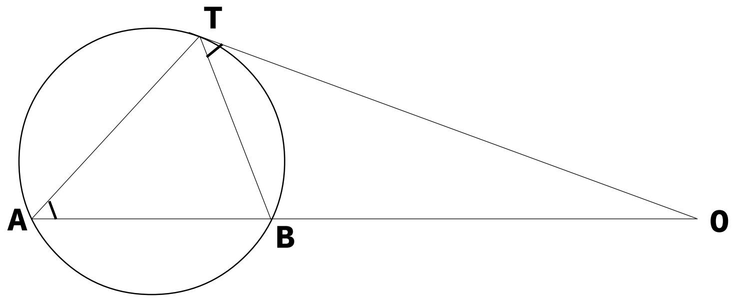
Hence AOD, COB triangles are similar

$$\frac{AO}{CO} = \frac{OD}{OB}$$

AO . OB = CO . OD

Co Theorem

If from an external point a secant and tangent are drawn to a circle, the rectangle contained by the whole secant and the part of it outside the circle is equal to the square on the tangent.



Let OBA be a secant and OT a tangent drawn to the circle from the point O.

Given : **OBA is a secant and OT is a tangent.**

To Prove : **OA . OB = OT^2**

Construction : Join BT, AT

Proof : In triangles OAT, OTB.

$\angle AOT = \angle BOT$

$\angle OAT = \angle OTB$ (angles in alternate segment)

Therefore third angle in each are equal

Hence OAT, OTB triangles are similar

$$\frac{OA}{OT} = \frac{OT}{OB}$$

OA . OB = OT^2

Exercise 7

1 **O is the centre of a circle of radius 6 cm. A secant PXY drawn from a point P outside the circle meets the circle at X and Y. If OP = 10 cm find the length of the tangent drawn from the point P to the circle. If PX = 5 cm, find the length of XY.**

2 **A tangent drawn to a circle from a point P is PT. PQR is a secant to the circle. If PQ = 4 cm and PT = 8 cm, find the length of QR.**

3 In an acute angled triangle ABC, altitudes BD and CE intersect at H.
Prove that BH : HD = EH : HC

4 The length of a bridge which connects banks of a river is 100m. A foot path designed above the bridge looks like an arc of a circle. Two pillars A and B at the end points of the bridge bear the foot path.
If the highest point C in the foot path is 20 m from the bridge, find the radius of the arc.

5 TA and TB are two tangents drawn from a point T to a circle. OT intersects AB at X.
Prove that (i) AX . XB = OX . XT
(ii) $OX . OT = OA^2$

6. A, B are centres of two circles, C_1 and C_2, which do not intersect each other. A third circle with centre O intersect circle C_1 at C, D and C_2 at E, F.
If two straight lines CD, EF produced to meet at P, prove that the lengths of the tangents drawn from P to C_1 and C_2 are equal.

7. AB and AC are two chords of a circle. A straight line parallel to the tangent at A, intersects AB and AC at D and E respectively.
Prove that AB . AD = AC . CE.

8. PQ and PR are two chords of a circle. Another chord PS of this circle intersects QR at T.
Prove that $PS . PT = PQ^2$

8. Pythagoras's Theorem and its extensions

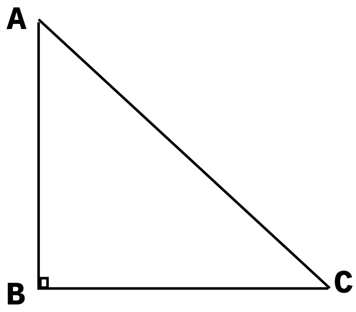
Theorem of Pythagoras

In a right angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the other two sides.

If ABC is a right angled triangle and

$\angle b = 90^0$,

Then $AC^2 = AB^2 + BC^2$



Converse of the above theorem

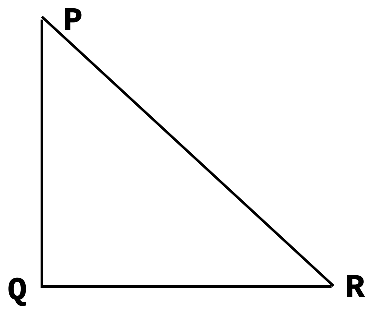
Theorem

If the square described on one side of the triangle equal to the sum of the squares described on the other two sides, then the angle contained by these two sides is a right angle.

In the triangle PQR if

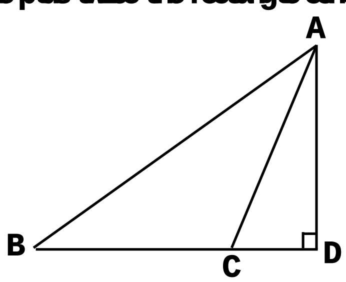
$PQ^2 + QR^2 = PR^2$

then $\angle PQR = 90^0$



Theorem

In an obtuse angled triangle the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle plus twice the rectangle contained by either of these sides and the projection on it of the other.



Given : **ABC is a triangle with an obtuse angle at C. CD is the projection of AC upon BC.**

To Prove : $AB^2 = BC^2 + CA^2 + 2CB \cdot CD$

Proof : ABD is a right angled triangle.

$AB^2 = BD^2 + AD^2$ (Pythagoras's theorem)

$= (BC + CD)^2 + AD^2$

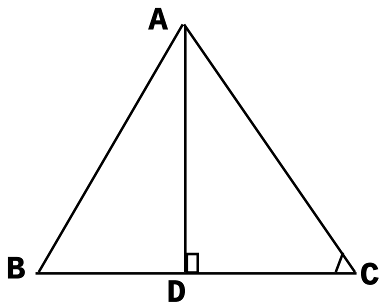
$= BC^2 + 2\ BC \cdot CD + CD^2 + AD^2$

$= BC^2 + 2\ BC \cdot CD + AC^2$

$= BC^2 + AC^2 + 2\ BC \cdot CD$

Theorem:

In any triangle the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing the acute angle less twice the rectangle contained by one of those sides and the projection on it of the other.



Given : ABC is a triangle with an acute angle at C. CD is the projection of AC upon BC.

To prove : $AB^2 = CA^2 + CB^2 - 2\ CB \cdot CD$

Proof : ABD is a right angled triangle.

$AB^2 = AD^2 + BD^2$

$= AD^2 + (BC - CD)^2$

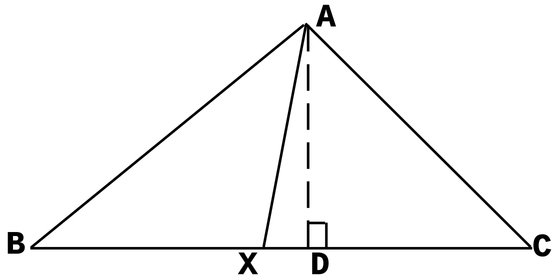
$= AD^2 + BC^2 - 2\ BC \cdot CD + CD^2$

$= AD^2 + CD^2 + BC^2 - 2\ BC \cdot CD$

$= AC^2 + BC^2 - 2\ BC \cdot CD$

Apollonius' Theorem:

The sum of the squares on two sides of a triangle is equal to twice the sum of the squares on half the third side and twice the square on the median which bisects that side.



Given : ABC is a triangle. BX = XC. AX is a median.

To prove : **AB**^{2} + AC^2 = $2BX^2 + 2AX^2$

Construction : Draw AD perpendicular to BC.

Proof : Of the angles AXB, AXC, one is obtuse and the other is acute.

Let the angle AXB is obtuse.

In triangle AXB,

$AB^2 = AX^2 + BX^2 + 2\,AX\,.\,XD$ (1)

In triangle AXC,

$AC^2 = AX^2 + XC^2 - 2\,AX\,.\,XD$ (2)

Adding (1) and (2),

$AB^2 + AC^2 = 2\,AX^2 + 2\,BX^2$ (since BX = XC)

Excercise 8

1 **ABC is an equilateral triangle. O is the mid point of BC. Prove that 3 BC**^{2} = 4 OA^2.

2. ABCD is a square of side 12 cm length. Prove that the area of the square described on the diagonal BD is twice the area of the square ABCD.

3. PQR is a triangle right angled at Q. Mid points of QR and PQ are X and Y respectively. Show that $6PR^2 = 4(PX^2 + RY^2)$

4. If from any point O within a triangle ABC, perpendiculars OX, OY, OZ are drawn to BC, CA, AB respectively. Show that $AZ^2 + BX^2 + CY^2 = AY^2 + CX^2 + BZ^2$

5. In a triangle ABC, AD is drawn to perpendicular to BC. Let p denote the length of AD.

   (i) If a = 25 cm, p = 12 cm, BD = 9 cm find b, c

   (ii) If b = 82 cm, c = 1 cm, BD = 60 cm; find p and c and prove that $\sqrt{b^2 - p^2} + \sqrt{c^2 - p^2} = a$ (AB = c, BC = a, CA = b)

6. ABC is a triangle right angled at C, and p is the length of the perpendicular from C on AB. By expressing the area of the triangle in two ways, show that pc = ab.

   Hence deduce $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$ (BC = a, CA = b, AB = c)

7. In triangle ABC, AD is a median. Point X is on the side BC, so that BX = XD and $A\hat{X}B = 90^0$. Prove that $4(AC^2 - AD^2) = BX^2$

8. ABD is a triangle right angled at A. Point C is in the side BD so that 2 BC = CD and $A\hat{C}D = 90^0$. If CT is a median of the triangle ACD prove that $2(CT^2 + AT^2) = AD^2$

9. In a triangle ABC, the points E and D are taken on BC such that BE = ED = DC. Prove that $AB^2 + AE^2 = AC^2 + AD^2$

10. In a Δ ABC, D is the mid point of BC. Find the length of the median AD when AB = 4 cm, BC = 5 cm and AC = 6 cm.

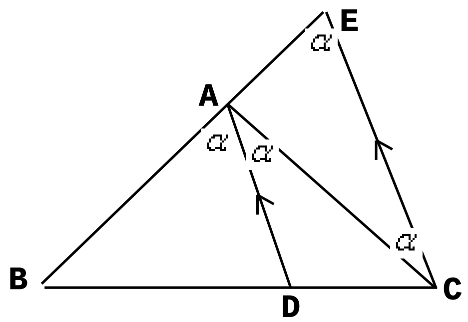
9. Bisector Theorem

Theorem

(a) **The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle bisected.**

(b) **The external bisector of an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle bisected.**

(a)



(a)

Given : **AD is the internal bisector of the angle BAC of the triangle ABC and meets BC at D.**

To prove : $\frac{AB}{AC} = \frac{BD}{CD}$

Construction : **From C draw CE parallel to DA to meet BA produced at E.**

Proof : $AD \parallel CE$

$\angle DAC = \angle ACE$ (Alternate angles)

$\angle BAD = \angle AEC$ (Corresponding angles)

$\angle DAC = \angle BAD$ (Given)

Therefore $\angle ACE = \angle AEC$

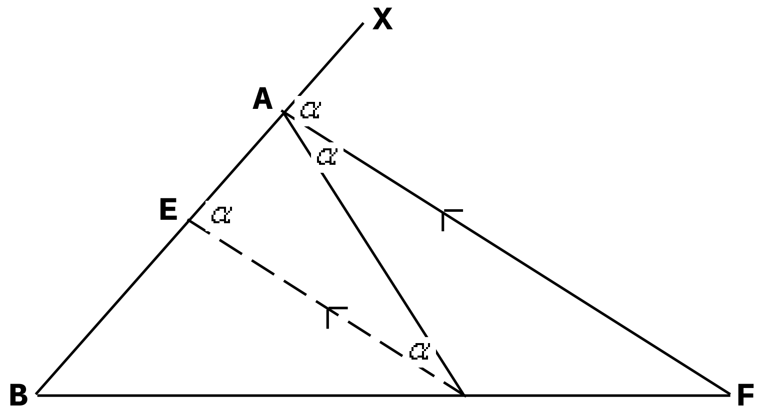
$AE = AC$

Since AD is parallel to EC

$$\frac{BA}{AE} = \frac{BD}{DC} \quad \text{and} \quad AE = AC$$

$$\frac{AB}{AC} = \frac{BD}{CD}$$

(b)



Given : **AF is the external bisector and meets BC produced, at F.**

To prove : $\frac{AB}{AC} = \frac{BF}{CF}$

Construction : **From C draw CE parallel to FA to meet BA at E.**

Proof : $CE \parallel FA$

$\angle ECA = \angle CAF$ (Alternate angles)

$\angle CEA = \angle FAX$ (Corresponding angles)

But $\angle CAF = \angle FAX$

Therefore $\angle ECA = \angle CEA$

$AE = AC$

In triangle BAF, $CE \parallel FA$

$\frac{BF}{CF} = \frac{BA}{AE}$ and since $AE = AC$

$\frac{BF}{CF} = \frac{BA}{AC}$

Exercise 9

1 In a quardrilateral PQRS, PQ // SR. PR and QS intersect at T. Prove that

(i) $\Delta PQT \,//\, \Delta SRT$

(ii) $\frac{PR}{PT} = \frac{QS}{QT}$

2 In a triangle ABC, the interior and the exterior bisectors of the angle meet BC at X and Y respectively. If AB = 7.2 cm, AC = 5.4 cm, BC = 3.5 cm.

(i) Prove that $BX : XC = 4 : 3$

(ii) Find the ratio $BY : YC$

3 PS is a median of triangle PQR. Bisectors of the angle PSQ, and PSR meet PQ and PR at L and M respectively. Prove that LM // QR.

4 In a quadrilateral ABCD, the bisectors of the angles BAC and DAC meet BC and CD at L and M respectively. Prove that LM // BD.

5 If I is the in-centre of the triangle PQR, and if PI is produced to meet BC at X, show that

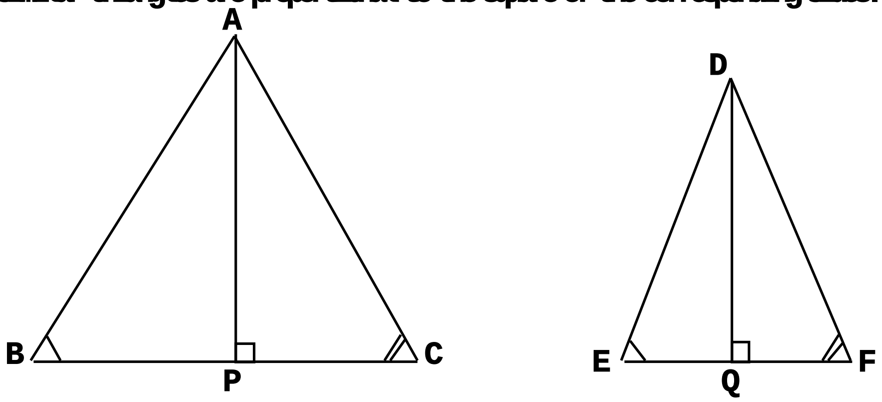
$PI : IX = PQ + PR : QR$

6 AD is a median of a triangle ABC. E is a point on produced AD. The bisector of $B\hat{D}E$ meets produced AB at H and the bisector of $C\hat{D}E$ meets produced AC at K. Show that HK // BC.

10. Area (Similar Triangles)

Theorem

The areas of similar triangles are proportional to the square of the corresponding sides.



Given : **Triangles ABC, DEF are similar.**

To prove : $\frac{\text{Area of triangle ABC}}{\text{Area of triangle DEF}} = \frac{BC^2}{EF^2}$

Construction : **Draw AP perpendicular to BC and DQ perpendicular to EF.**

Proof : **In triangles APB, DQE**

$\angle ABP = \angle DEQ$ (given)

$\angle APB = \angle DQE$ $(= 90^0)$

Therefore third angle in each are equal.

Triangles APB, DQE are similar.

Δ's APB, DQE ///

$\frac{AP}{DQ} = \frac{AB}{DE}$ **But** $\frac{AB}{DE} = \frac{BC}{EF}$

Therefore $\frac{AP}{DQ} = \frac{BC}{EF}$

Now, $\frac{\text{Area of triangle ABC}}{\text{Area of triangle DEF}} = \frac{\frac{1}{2}BC \times AP}{\frac{1}{2}EF \times DQ}$

$= \frac{BC}{EF} \times \frac{AP}{DQ}$

$= \frac{BC}{EF} \times \frac{BC}{EF}$

$= \frac{BC^2}{EF^2}$

Excercise 10

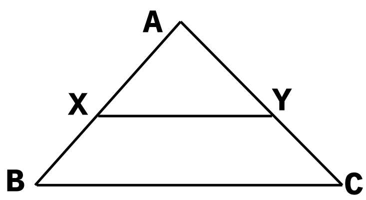
1 **ABC is a right angled triangle, right angled at A. AO is perpendicular to BC.**
Prove that $\Delta BAD : \Delta ACD = BA^2 : AC^2$

2 **In trapezium ABCD, AB is parallel to BC. AC and BD interesect at O. If AO =** $\frac{1}{4}$ **AC, prove that** $\Delta AOB = \frac{1}{9}\Delta COD$

3 **ABC is an isoceles triangle right angled at A. Outside the triangle ABC, ABD and BCE are two equalateral triangles on AB and BC respectively.**
Prove that $\Delta ABD : \Delta BCE = 1 : 2$

4 **ABCD is a trapezium. AB is parallel to CD. If diagonals intersect at O and AB = 2cm find the ratio between** Δ **ADB and** Δ **COD.**

5 **ABC is a triangle. XY is parallel to BC. If** $\Delta AXY : \mathit{fig}\ XBCY = 4:5$ **show that** $AX : XB = 2:1$.



6 **ABC is an acute angled triangle. BD, CE are altitudes. BD, CE intersect at X.**
Fill in the blanks.

a) $\Delta ABD : \Delta ACE = BD^2 :$ ....

b) $\Delta BXE : \Delta CXD =$ .... $: CD^2$

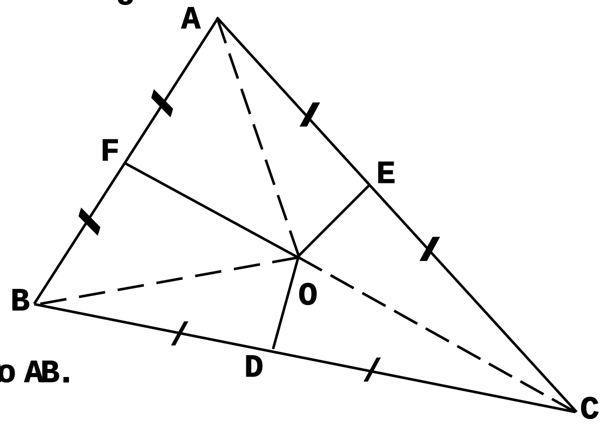
c) $\Delta ABD :$ .... $= AD^2 : AE^2$

d) $\Delta BXE :$ .... $= BX^2 : CX^2$

11. Concurrencies connected with a Triangle

1. Perpendicular bisectors of the sides

The perpendicular bisectors of the three sides of a triangle are concurrent.



Given : OD and OE are the perpendicular bisectors of the sides BC and CA respectively of a triangle ABC. They intersect at O. Let F be the mid point of BA and join OF.

To prove : OF is perpendicular to AB.

Construction : Join OA, OB, OC.

Proof : In triangles BDO, CDO

BD = CD (given)

CD = CD (common)

$\angle BDO = \angle CDO$ (=90^0, given)

$\Delta BDO \equiv \Delta CDO$ (SAS)

Therefore OB = OC ----------------(1)

Similarly $\Delta OCE \equiv \Delta OAE$

Therefore OC = OA -----------------(2)

From (1) and (2) OA = OB.

In triangles OAF, OBF

OA = OB (proved)

OF = OF (common)

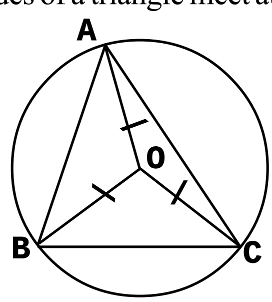
AF = FB (given)

$\Delta OAF \equiv \Delta OBF$ (SSS)

$\angle OFA = \angle OFB = 90^0$

Therefore OF is perpendicular to AB.

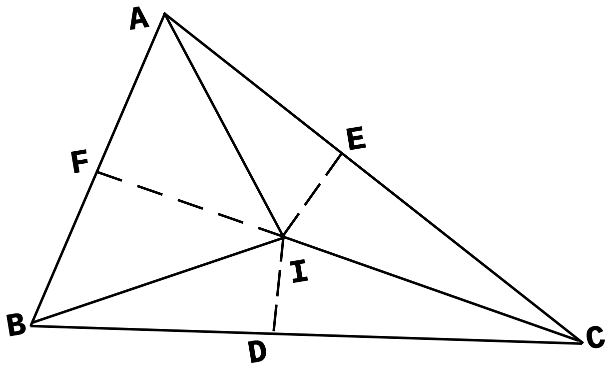
Hence, the three perpendicular bisects of the sides of a triangle meet at a point.



In this diagram, OA=OB=OC. O is said to be the *circumcentre* **of the triangle ABC. The circle is called** *circumcircle* **of the triangle ABC.**

2. Bisectors of the angles

The bisectors of the three angles of a triangle are concurrent.



Given : IB and IC are the bisectors of the angles ABC and ACB of a triangle ABC. Join AI.

To prove : AI bisects the angle BAC.

Construction : Draw perpendiculars ID, IE and IF to BC, CA and AB respectively.

Proof : In triangles BDI, BFI

$\angle IBD = \angle IBF$ (given)

$\angle BDI = \angle BFI$ (=90^0, given)

BI = BI (common)

$\Delta BDI \equiv \Delta BFI$ (RHS)

$\therefore$ ID = IF ——————— (1)

Similarly it may be proved that

$\Delta CDI \equiv \Delta CEI$

$\therefore$ ID = IE ——————— (2)

From (1) and (2), IE = IF.

In triangles AEI, AFI

IE = IF (proved)

IA = IA (common)

$\angle IEA = \angle IFA$ (=90^0)

$\Delta AEI \equiv \Delta AFI$

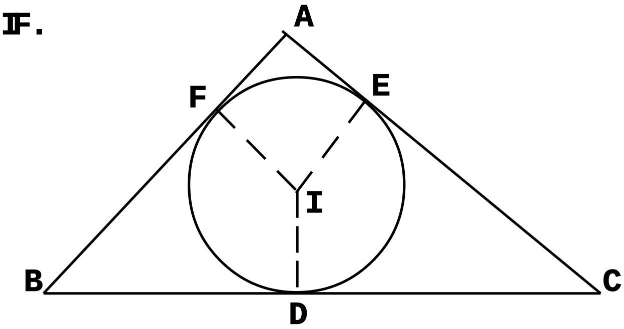
Therefore $\angle EAI = \angle FAI$

ie. IA is the bisector of the angle BAC.

Hence, the bisectors of three angles of a triangle are concurrent.

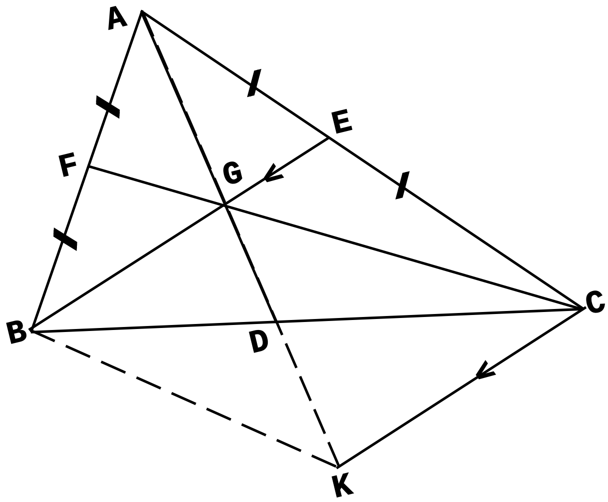
It is said to be ***incentre*** and the circle is called ***inscribed circle.***

radius = ID = IE = IF.



3. Medians

The three medians of a triangle are concurrent.



Given : E and F are the mid points of the sides AC, AB of a triangle ABC. BE and CF meets at G. Join A G and produce it to meet BC at D.

To prove: BD = DC

Construction : Through C draw CK parallel to EB; produce AD to meet CK at K. Join BK.

Proof : In the triangle AKC,

AE = EC (given)

EG // CK (construction)

Therefore, AG = GK ———————— (1)

In the triangle A B K

AF = FB (given)

AG = GK (proved)

Therefore, FG // BK ———————— (2)

In the quadrilateral CGBK.

CK // GB

BK // GC

Thefefore CGBK is a parallelogram.

The diagonals of a parallelogram bisect each other.

BD = DC and GD = DK.

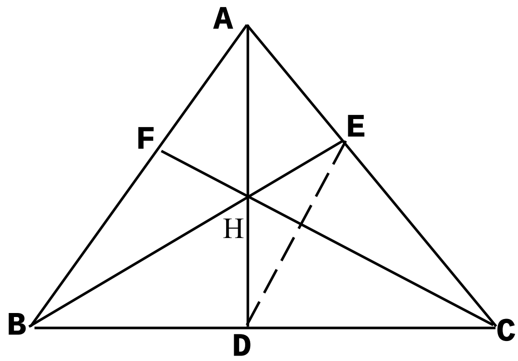
BD = DC means AD is a median of the triangle ABC.

Hence the three medians of a triangle meet at a point.

The point (G) of intersection of three medians is called *centroid* of the triangle.

4. Altitudes

The perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.



Given : **Let AD and BE be the perpendiculars drawn from A and B to the opposite sides and let them intersect at H. Join CH and produce it to meet AB at F.**

To prove : **CF is perpendicular to AB.**

Construction : **Join DE.**

Proof : **In the quadrilateral DCEH**

$\angle CDH + \angle CEH = 90^0 + 90^0 = 180^0$

DCEH is a cyclic quadrilateral.

$\angle DEC = \angle DHC$ **(angle in the same segment)** ———— **(1)**

$\angle DHC = \angle AHF$ **(the vertically opposite angles)**

since $\angle AEB = \angle ADB$ **(=90**^{0})

AEDB is cyclic quadrilateral.

$\angle ABD = \angle DEC$ (exterior angle = interior opposite angle) ——— (2)

From (1) and (2) $\angle ABD = \angle DHC$

Therefore, BDHF is a cyclic quadrilateral.

$\angle HDB + \angle HFB = 180^0$

$\angle HFB = 90^0$

i.e., CF is perpendicular to AB.

Hence the three perpendiculars AD, BE and CF meet at H.
H is called ***orthocentre***.

Exercise

1. **The sides AB, AC of a triangle ABC are produced. Show that the bisectors of the exterior angles of B and C and the bisector of the interior angle of A are concurrent.**

12. Answers

12.1 Answers for exercise 1.1

1 $4a^2+12ab+9b^2$

2 $9a^2-24ab+16b^2$

3 $x^2+2+\frac{1}{x^2}$

4 $4x^2y^2+20xyz+25z^2$

5 $\frac{1}{a^2}+\frac{2}{ab}+\frac{1}{b^2}$

6 $x^2-2+\frac{1}{x^2}$

7. $\frac{a^2}{4}-2+\frac{4}{a^2}$

8 $\frac{1}{a^2}-\frac{4}{ab}+\frac{4}{b^2}$

9 $16x^2y^2-24xyz+9z^2$

10. $a^3+6a^2b+12ab^2+8b^3$

11 $8a^3-12a^2b+6ab^2-b^3$

12. $27a^3+54a^2b+36ab^2+8b$

13. 27 $x^3+3x+\frac{3}{x}+\frac{1}{x^3}$

14. $x^3-3x+\frac{3}{x}-\frac{1}{x^3}$

15. $a^3b^3-6a^2b^2c+12abc^2-8c^3$

16. $\frac{1}{a^3}+\frac{3}{a^2b}+\frac{3}{ab^2}+\frac{1}{b^3}$

17. $\frac{1}{a^3}-\frac{6}{a^2b}+\frac{12}{ab^2}-\frac{8}{b^3}$

18. $8x^3y^3-36x^2y^2z+54xyz^2-27z^3$

19. $a^2+b^2+c^2+2ab+2bc+2ca$

20. $a^2+b^2+c^2+2ab-2bc-2ca$

21. $a^2+b^2+c^2-2ab-2bc-2ca$

22. $a^2+b^2+c^2-2ab-2bc-2ac$

23. $a^2+4b^2+c^2-4ab-4bc+2ac$

24. $a^2+b^2+4c^2-2ab+4bc-4ac$

25. (i) 1 030 301 (ii) 7 762 392 (iii) 64 481 201 (iv) 997 002 299

26. (a) 4 000 000 (b) 1

27. (a) 1 000 000 (b) 8

29. (a) a^2 -2 (b) 8 -3a

30. 316

31. $-\frac{1}{27}$

32. -340

34. 14

35. 27

36. 427

12.2.1 Answers for exercise 2.1

1 $(x-3)(x+2)$

2 $(x+12)(x-8)$

3 $(x+6)(x-1)$

4 $(x-6)(x+2)$

5	$(x+7)(x-6)$	6	$(x-3)(x-6)$
7.	$(2x+3)(x+1)$	8	$(2x-3)(x-1)$
9	$(2x-1)(x+3)$	10.	$(2x+1)(x-3)$
11	$(2+3x)(5+4x)$	12.	$(5-2x)(3-x)$
13.	$3(3x+8)(2x-9)$	14.	$(2x-9)(3x-14)$
15.	$(2x-3y)(x-y)$	16.	$(3x+2y)(2x-3y)$
17.	$(2x+y)(2x+3y)$	18.	$(2a-b)(a-13b)$
19.	$(8xy-3)(5xy+8)$	20.	$(8x+5y)(4x-7y)$
21.	$a(8a+5b)(3a-4b)$	22.	$a(3a+2b)(6a-5b)$
23.	$(a-2)(a-1)(a-8)(a+5)$	24.	$(a+b+c-7)(a+b+c+4)$
25.	$(2x+2y-9)(x+y+3)$	26.	$(x+8y)(x+3y)$
27.	$(x-a+2)(x+a-1)$	28.	$(x+a-2)(x-a+1)$
29.	$(x-a)\left(x-\frac{1}{a}\right)$	30.	$(x+a+b)(x+a-b)$
31.	$(ax-1)(x+b)$	32.	$(x+3a-b)(x-2a+b)$
33.	$(2a^2+ab-2b^2)(2a^2-5ab-2b^2)$	34.	$(a+12b)(12a-b)$
35.	$(5x-y)(5y-x)$		

12.2.2 Answers for Exercise no. 2.2

1	$(x-2y)(x+2y)$	2	$x(x-1)(x+1)$
3	$\left(x-\frac{1}{x}\right)\left(x+\frac{1}{x}\right)$	4	$x(x-1)(x+1)(x^2+1)$
5	$(2-3a)(2+3a)$	6	$(a-7b)(a-b)$
7.	$(4-a-b)(4+a+b)$	8	$(3-a+b)(3+a-b)$
9	$3a(2a-b)(2a+b)$	10.	$(1-a+b)(1+a-b)$
11	$(1-a-b)(1+a+b)$	12.	$(x+y)(x-y-1)$
13.	$(x-y)(x+y-1)$	14.	$(x+y)(x-y+1)$
15.	$(x-y)(x+y+1)$	16.	$(a-b)(a+b-4)$
17.	$(a-2-b)(a-2+b)$	18.	$(b+c)(a-b-c)$
19.	$(a-b)(a+b+1)$	20.	$(x^2-y^2-xy)(x^2+y^2+xy)$
21.	$(x^2+y^2-xy)(x^2+y^2+xy)$	22.	$(a^2+3b^2-ab)(a^2+3b^2+ab)$
23.	$(x-2y-z)(x-2y+z)$	24.	$(2a+b-x)(2a+b+x)$
25.	$(x^2+x+1)(x^2-x+1)$	26.	$(2a+3b-ab)(2a+3b+ab)$

27.	199	28.	8800
29.	95.2	30.	24
31.	9901	32.	186
33.	144	34.	1
35.	0.25	36.	9991

12.2.3 Answers for Exercise no. 2.3

1. $(a+2b)(a^2-2ab+4b^2)$
2. $(3a-b)(9a^2+3ab+b^2)$
3. $(5a-4b)(25a^2+20ab+16b^2)$
4. $(2ab-c)(4a^2b^2+2abc+c^2)$
5. $\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)$
6. $\left(x-\frac{1}{x}\right)\left(x^2+1+\frac{1}{x^2}\right)$
7. $\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{a^2}-\frac{1}{ab}+\frac{1}{b^2}\right)$
8. $\left(\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a^2}+\frac{1}{ab}+\frac{1}{b^2}\right)$
9. $(a+b+c)(a^2+b^2+c^2-ab-ac+2bc)$
10. $(a+b-c)(a^2+b^2+c^2-ab-2bc+ac)$
11. $(a-b+c)(a^2+b^2+c^2+ab-ac-2bc)$
12. $(x+2y)(7x^2-8xy+4y^2)$
13. $2a(a^2+3b^2)$
14. $2b(3a^2+b^2)$
15. $(a+b)(2a+2b+1)(4a^2+4b^2+8ab-2a-2b+1)$
16. $(x-y)(x+y)(x^4+x^2y^2+y^2)$
17. $(x^2+y^2)(x^4-x^2y^2+y^4)$
18. $(x-\sqrt{3})(x+\sqrt{3})(x^4+3x^2+9)$
19. (a) $(a+b+c)(a^2+b^2+c^2+2ab-ac-bc)$
    (c) $(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$
    (i) $(x+y-z)(x^2+y^2+z^2-xy+xz+yz)$
    (ii) $(2x+y+z)(4x^2+y^2+z^2-2xy-2xz-yz)$
    (iii) $(x+2y+3z)(x^2+4y^2+9z^2-2xy+2xz+6yz)$
    (iv) $(a-4b)(a^2+7b^2+4ab)$
    (v) $(2a+b-1)(4a^2+b^2+1-2ab+b+2a)$

21. (i) $3(a-b)(b-c)(c-a)$

(ii) $6(2x-3y)(3y-4z)(2z-x)$

(iii) $abc(b-c)(c-a)(a-b)$

(iv) $3(x-3y)(3y-4z)(4z-x)$

12.2.3 Answers for Exercise no. 3.1

1.	$\frac{x+1}{2}$	2.	$\frac{-1}{(x+4)(x+5)}$	3.	$\frac{x+4}{(x-3)(x+2)(x+5)}$
4.	$\frac{3}{x-3}$	5.	$\frac{x^2+3}{x+1}$	6.	0
7.	$\frac{-x-2}{(2x-1)(2x+1)(x-1)}$	8.	$\frac{4(a+1)}{(a-5)(a-4)(a+3)}$	9.	$\frac{a^2+4}{a^2-4}$
10.	$\frac{2}{x^2-1}$	11.	$\frac{a}{a-c}$	12.	2
13.	$\frac{a^2-1}{a^2-4}$	14.	$\frac{a^2-ab+b^2}{a^2-b^2}$	15.	$\frac{1}{2a(a+b)}$
16.	$\frac{a+1}{a-1}$	17.	$\frac{x(x-y+z)}{z(x+y+z)}$	18.	$\frac{ab}{a^2+b^2}$
19.	$z=\frac{x^4+x^2+1}{x(x^2+1)}$	20.	$\frac{2t}{1+t^2}$	21.	$\frac{2a}{1-a^2}$
22.	$\frac{7c+4}{5c+1}$				

12.4.1.1 Answers for Exercise no. 4.1.1

1.	$-\frac{3}{2}$ or -1.5	2.	4	3.	37	4.	(-10)
5.	$\frac{156}{23}$	6.	-10	7.	8	9.	-3
10.	(-2)						

12.4.1.2 Answers for Exercise no. 4.1.2

1. $x=0$ or $x=\frac{2}{3}$ 2. -1, -3

3	2, 6	4	$-\frac{1}{2}$, 3
5	$-\frac{1}{2}$, 2	7.	-12, 9
8	$5/2$, 6	9	-1, $3/2$
10.	0, 4	11	-17, -1
12.	-3, $7/2$	13.	$3+\sqrt{14},\ 3-\sqrt{14}$
14.	$\frac{-7-\sqrt{89}}{4},\ \frac{-7+\sqrt{89}}{4}$	15.	$\frac{+3-\sqrt{65}}{4},\ \frac{+3+\sqrt{65}}{4}$
16.	$\frac{5-\sqrt{29}}{2},\ \frac{5+\sqrt{29}}{2}$		

12.4.1.3 Answers for Exercise no. 4.1.3

1	-4, -3, -2, -1	2	2, 7
3	1, 2, -3, $\pm\sqrt{7}$	4	$-\frac{3}{4},\ -\frac{2}{3}$
5	$-\frac{62}{9},\ 2$	6.	4, 36
7.	20, 125	8.	$\frac{1}{4}$, 4
9	-8, 1	10.	$\frac{1}{27},\ 8$
11	$-\frac{2}{5},\ 1,\ \frac{-36-2\sqrt{15}}{11},\ \frac{-36+2\sqrt{15}}{11}$	12.	$-\frac{4}{5},\ \frac{9}{4}$
13.	$-6,\ 0,\ \frac{-4-3\sqrt{2}}{2},\ \frac{-4+3\sqrt{2}}{2}$	14.	$-\frac{1}{2},\ 2,\ \frac{-17+\sqrt{305}}{4},\ \frac{-17-\sqrt{305}}{4}$
15.	1	16.	$\frac{1}{3},\ 3$
18.	$\frac{1}{3},\ 1,\ 3$	19.	$\frac{1}{3},\ 1,\ 3$
20.	$-\frac{1}{2},\ -\frac{1}{4},\ 2,\ 4$	21.	$-1,\ -\frac{1}{2},\ 1,\ 2$
22.	-2, 0	23.	-1, 2

24. $-\frac{1}{2}, \frac{1}{2}$

25. -2

26. 3, $\frac{83}{17}$

27. 3, 7

28. -3, 3

29. -3, 2, 7

30. $\frac{1}{2}$, 2

31. $\frac{-5+\sqrt{5}}{2}, \frac{-5-\sqrt{5}}{2}$

12.4.2 Answers for Exercise no. 4.2

1. $x=-2, \ y=3$
2. $x=1, \ y=-2$
3. $x=3, \ y=-1$
4. $x=2, \ y=-1$
5. $x=5, \ y=\frac{1}{2}$
6. $x=\frac{1}{7}, \ y=\frac{1}{4}$
7. $x=\frac{1}{2}, \ y=-\frac{1}{3}$
8. $x=-1, \ y=-\frac{2}{5}$
9. $x=11, \ y=\frac{1}{2}$
10. $x=2, \ y=-1$
11. $x=a-b, \ y=b-a$
12. $x=\frac{1}{4}, \ y=-1$
13. $x=5, \ y=9$
14. $x=-\frac{1}{2}, \ y=\frac{1}{4}$
15. $x=\frac{(a^2+b^2)}{2ab}, \quad y=\frac{-(a^2-2ab-b^2)}{2ab}$

12.4.3 Answers for Exercise no. 4.3

1. $\left.\begin{matrix} x=-\frac{1}{2} \\ y=0 \end{matrix}\right\}$ $\left.\begin{matrix} x=-1 \\ y=-1 \end{matrix}\right\}$

2. $\left.\begin{matrix} x=-7 \\ y=-4 \end{matrix}\right\}$ $\left.\begin{matrix} x=7 \\ y=3 \end{matrix}\right\}$

3. $\left.\begin{matrix} x=5 \\ y=-\frac{3}{5} \end{matrix}\right\}$ $\left.\begin{matrix} x=7 \\ y=-3 \end{matrix}\right\}$

4. $\left.\begin{matrix} x=-4 \\ y=8 \end{matrix}\right\}$ $\left.\begin{matrix} x=3 \\ y=1 \end{matrix}\right\}$

5. $\left.\begin{matrix} x=8 \\ y=\frac{1}{2} \end{matrix}\right\}$ $\left.\begin{matrix} x=\frac{1}{3} \\ y=12 \end{matrix}\right\}$

6. $\left.\begin{matrix} x=\frac{1}{3} \\ y=\frac{3}{2} \end{matrix}\right\}$ $\left.\begin{matrix} x=\frac{1}{5} \\ y=\frac{13}{10} \end{matrix}\right\}$

7. $\left.\begin{matrix} x=4 \\ y=3 \end{matrix}\right\}$ $\left.\begin{matrix} x=4 \\ y=-3 \end{matrix}\right\}$ $\left.\begin{matrix} x=\sqrt{2} \\ y=-3 \end{matrix}\right\}$ $\left.\begin{matrix} x=-\sqrt{2} \\ y=-3 \end{matrix}\right\}$

9. $\left.\begin{matrix} x=\frac{1}{5} \\ y=\frac{3}{5} \end{matrix}\right\}$ $\left.\begin{matrix} x=1 \\ y=3 \end{matrix}\right\}$ $\left.\begin{matrix} x=-6\pm\sqrt{143} \\ y=6\frac{\pm\sqrt{143}}{2} \end{matrix}\right\}$

10. $\left.\begin{matrix} x=-\frac{5}{2} \\ y=-\frac{5}{2} \end{matrix}\right\}$ $\left.\begin{matrix} x=1 \\ y=1 \end{matrix}\right\}$ $\left.\begin{matrix} x=-1\pm\sqrt{21} \\ y=1\frac{\pm\sqrt{21}}{2} \end{matrix}\right\}$

13. $\left.\begin{matrix} x=-6 \\ y=\frac{5}{8} \end{matrix}\right\}$ $\left.\begin{matrix} x=3 \\ y=4 \end{matrix}\right\}$

14. $x=\frac{84}{25}, \quad y=-25$

15. $\left.\begin{matrix} x=\frac{16}{9} \\ y=-\frac{3}{4} \end{matrix}\right\}$ $\left.\begin{matrix} x=3 \\ y=-\frac{5}{3} \end{matrix}\right\}$

17. $x=\frac{10}{3},\ y=\frac{106}{33},\ z=\frac{52}{33}$

18. $x=1, \quad y=2, \quad z=-6$

21. $\left.\begin{matrix} x=4 \\ y=3 \\ z=5 \end{matrix}\right\}$ $\left.\begin{matrix} x=-6 \\ y=1 \\ z=-3 \end{matrix}\right\}$

Answers for Exercise no. 12.5

1 (a) 6 (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) $\frac{1}{5}$

2 (a) $\frac{1}{100}$ (b) $\frac{1}{100}$ (c) $\frac{3}{2}$

3 (a) 216 (b) 13 (c) 5

4 36

5 (a) 4 (b) $\frac{1}{4}$ (c) $\frac{3}{7}$ (d) 6 (e) 4, -2 (f) 0, 1

6 (a) 4 (b) 4 (c) 4 (d) 3

7. (a) 1 (b) -1 (c) $\frac{3}{2}$ (d) 2

8 (a) $\frac{1}{6}$ (b) $\frac{4}{3}$ (c) -2

9 (a) -2 (b) $\frac{1}{3}$